6 Algebraic Models

What You'll Learn

How to rearrange formulas, evaluate powers, and solve exponential equations

And Why

These skills can help you solve real-world problems in fields such as business, construction, design, medicine, science, and technology.

Key Words

- square root
- formula
- inverse operations
- base
- exponent
- power
- cube root
- rational exponent
- exponential equation

CHAPTER 6

Activate Prior Knowledge

Square Roots

Prior Knowledge for 6.1

The **square** of a number is the number multiplied by itself. Finding the **square root** is the **inverse operation** of squaring. For example, since $5^2 = 5 \times 5 = 25$ and $(-5)^2 = (-5) \times (-5) = 25$, both 5 and -5 are square roots of 25. We write $\sqrt{25} = 5$, and $-\sqrt{25} = -5$.

$\sqrt{25}$ is the positive square root of 25, while $-\sqrt{25}$ is the negative square root of 25. So, $\sqrt{25} = 5$ and $-\sqrt{25} = -5$.

Example

Materials

- scientific calculator

Using a TI-30X IIS scientific calculator, press: 3 [2nd] [x^2] 5 [)] [ENTER] If you are using a different calculator, refer to the user's manual.

Evaluate. Round to the nearest hundredth where necessary.

a) $\sqrt{36}$ **b)** $-\sqrt{100}$ **c)** $\sqrt{123}$ **d)** $3\sqrt{5}$

Solution

a) $\sqrt{36} = 6$ since $6^2 = 36$

b) $-\sqrt{100} = -10$ since $(-10)^2 = 100$

c) Use the square root key on a calculator: $\sqrt{123} \doteq 11.09$

d) Use the square root key on a calculator: $3\sqrt{5} \doteq 6.71$

CHECK ✓

1. Evaluate. Round to the nearest hundredth where necessary.

a) $\sqrt{49}$ **b)** $-\sqrt{64}$ **c)** $\sqrt{10}$ **d)** $-\sqrt{81}$

e) $2\sqrt{7}$ **f)** $-\sqrt{9}$ **g)** $3\sqrt{16}$ **h)** $\sqrt{\dfrac{8}{\pi}}$

For which parts did you use a calculator? Explain.

2. Integers whose square roots are also integers are called *perfect squares.*

a) Explain why 81 is a perfect square, but 82 is not.

b) Write the first 12 perfect square integers and their square roots.

3. The formula $T = 2\pi\sqrt{\dfrac{L}{9.8}}$ gives the time, T seconds, for one complete swing of a pendulum with length L metres. A clock pendulum is 22 cm long. Determine, to the nearest tenth of a second, the time it takes to complete one swing.

Solving Linear Equations

Use a balance model to solve linear equations in one variable. Perform the same operation on both sides of the equation until the variable is isolated on one side.

Example

Materials

- scientific calculator

Solve.

a) $2x + 1 = 7$ **b)** $7x + 3 = -2x + 9$

In a linear equation, all variables are raised to the first power ($x = x^1$).

Solution

a) $2x + 1 = 7$

Isolate $2x$ first, then solve for x.

$2x + 1 - 1 = 7 - 1$ Subtract 1 from each side.

$2x = 6$

$\frac{2x}{2} = \frac{6}{2}$ Divide each side by 2.

$x = 3$

To check, substitute $x = 3$ in $2x + 1 = 7$

L.S.	R.S.
$2(3) + 1$	7
$= 6 + 1$	
$= 7$	

L.S. = R.S., so the solution is correct.

b) $7x + 3 = -2x + 9$

Collect the variable terms on the left side, and the numbers on the right side.

$7x + 3 + 2x = -2x + 9 + 2x$ Add $2x$ to each side.

$9x + 3 = 9$

$9x + 3 - 3 = 9 - 3$ Subtract 3 from each side.

$9x = 6$

$\frac{9x}{9} = \frac{6}{9}$ Divide each side by 9.

$x = \frac{2}{3}$

CHECK ✓

1. Solve.

a) $x - 12 = -5$ **b)** $-3x = -54$ **c)** $5x - 3 = 12$ **d)** $-3x + 4 = 25$

2. Solve and check.

a) $13x + 8 = 6x + 22$ **b)** $3x - 11 = -2x + 9$ **c)** $-2x + 8 = -7x - 2$

Why should you always check your solution in the original equation?

3. The equation $T = 10d + 20$ gives the temperature, T degrees Celsius, at a depth of d kilometres below the surface of the Earth. Determine the depth of a mine shaft in which the temperature is 40°C. How do you know that your answer is correct? Explain.

Evaluating Powers with Integer Exponents

Prior Knowledge for 6.3

Positive integer exponent

$a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ factors}}$

Zero exponent

$a^0 = 1, a \neq 0$

Negative integer exponent

$a^{-n} = \frac{1}{a^n}, a \neq 0$

a^{-n} is the reciprocal of a^n.

Example

Materials

- scientific calculator

Evaluate.

a) 3^4 b) 2^{-5} c) $(-5)^0$ d) $\left(\frac{4}{5}\right)^{-2}$ e) 0.4^{-3}

Solution

a) $3^4 = (3)(3)(3)(3) = 81$

b) $2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

c) $(-5)^0 = 1$

d) $\left(\frac{4}{5}\right)^{-2} = \left(\frac{5}{4}\right)^2$ The reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$.

$= \frac{5}{4} \times \frac{5}{4}$ Raise $\frac{5}{4}$ to the exponent 2.

$= \frac{25}{16}$

The product of a non-zero number and its reciprocal is 1. The reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$ since $\frac{5}{4} \times \frac{4}{5} = 1$.

e) Use the exponent key on a calculator to obtain $0.4^{-3} = 15.625$

Press: 0.4 [^] [(–)] 3 [ENTER]

CHECK ✓

1. Evaluate without using a calculator.

a) 2^3 b) 4^3 c) $(-5)^2$ d) 3^{-2}

e) 8^0 f) $\left(\frac{1}{2}\right)^3$ g) $(-7)^{-1}$ h) $\left(\frac{3}{5}\right)^{-2}$

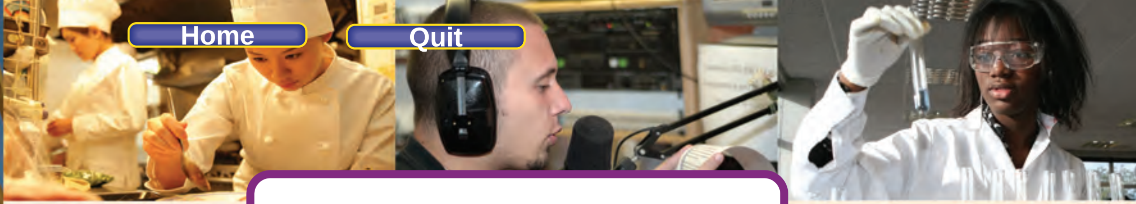
2. Evaluate with a calculator. Round to the nearest hundredth.

a) 0.95^7 b) 1.6^{-3} c) $200(1.04)^5$

d) $500(0.95)^{-3}$ e) $\left(\frac{2}{3}\right)^6$ f) $\left(\frac{5}{6}\right)^{-3}$

3. Explain the difference between the expressions in each pair and determine their values.

a) 3^2 and 2^3 b) 4^3 and $(-4)^3$ c) 5^2 and 5^{-2}



Learning with Others

At college, in an apprenticeship program, and in the workplace, people learn new skills and solve problems by working with others.

These strategies will help you succeed in team environments.

- Make sure everyone understands the situation in the task.
- Be creative with models and technology tools.
- Contribute ideas and information. Express your ideas clearly.
- Be open to, and listen actively to, the ideas of others.
- Analyse ideas and ask questions.
- Do your fair share of the work, and help your partners.
- Make sure everyone can explain the solution.

The goal is for each person to learn, so everyone needs to cooperate as a team to help each other understand the math thinking and communicate.

1. What is another strategy you would suggest for learning with others?

2. When working on this chapter, choose an *Investigate* and apply the strategies for learning with others.

3. Then, explain whether the strategies for learning with others were useful.

- Did the strategies help you understand the math? Did they help others understand the math? Include examples.
- Would you use some of the strategies in the next chapter? Explain your thinking.
- What suggestions would you give someone else for learning as a team?

4. Imagine being an apprentice, an employee, or a college student. How do you think you might use strategies for learning with others in this role?

Home Quit

6.1 Using Formulas to Solve Problems

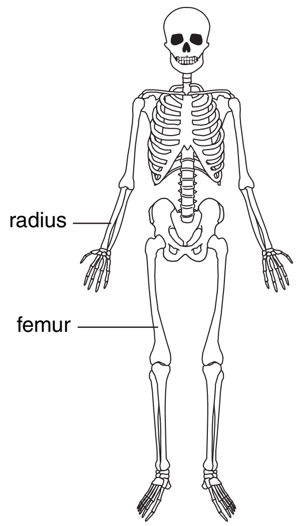
Forensic scientists and anthropologists use formulas to predict the height of a person from the lengths of their bones. They can use the radius bone or the femur bone.

Investigate

Estimating Height from the Lengths of Bones

Materials

- metre stick
- scientific calculator



Work with a partner.

These formulas give the height, h, of an adult in terms of the lengths of the radius bone, r, and femur bone, f.

Male	Female
$h = 3.65r + 80.41$	$h = 3.88r + 73.50$
$h = 2.24f + 69.09$	$h = 2.32f + 61.41$

All measurements are in centimetres.

- Predict the height of a female whose femur has length 40.6 cm.
- Predict the height of a male whose radius has length 28.1 cm.
- Have your partner measure the length of your radius and femur bones. Use each measure and the appropriate formula to estimate your height.
- Which formula gave the more accurate prediction of your height? Explain.

Reflect

- How do you think the formulas were obtained?
- Why is there a different set of formulas for males and females?
- What might account for the difference between your actual height and the heights predicted by the formulas?

Connect the Ideas

Formulas

A **formula** is a mathematical equation that relates two or more variables representing real-world quantities. Rules and procedures in many occupations are expressed as formulas.

Example 1 Substituting into a Formula

Materials
- scientific calculator

Pediatric nurses use *Young's formula*, $C = \frac{Ag}{g + 12}$, to calculate a child's dose of medicine, C milligrams, when the adult dose, A milligrams, and the child's age, g years, are known. Suppose the adult dose of a certain medication is 600 mg. Determine the corresponding dose for a 3-year-old child.

Solution

Substitute $A = 600$ and $g = 3$ in the formula $C = \frac{Ag}{g + 12}$.

$$C = \frac{(600)(3)}{3 + 12}$$
$$= \frac{1800}{15}$$
$$= 120$$

The child's dose is 120 mg.

Example 2 Using a Formula to Solve a Problem

Materials
- scientific calculator

A landscaper wants to estimate the cost of fertilizing a triangular lawn with side lengths 150 m, 200 m, and 300 m. One bag of fertilizer costs $19.98 and covers an area of 900 m^2.

She uses *Heron's formula* to determine the area of the lawn:

The area A of a triangle with side lengths a, b, and c is given by

$A = \sqrt{s(s - a)(s - b)(s - c)}$, where $s = \frac{a + b + c}{2}$.

Estimate the cost of fertilizing the lawn.

Plan your solution by working backward from what you are trying to find to what you are given. Write the solution by working forward from what you are given to what you are trying to find.

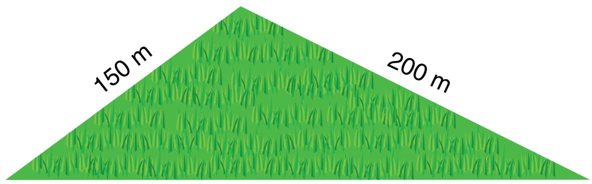
Solution

Plan the solution.

To find the:	We need to know the:
...cost of the fertilizer	...number of bags needed
...number of bags needed	...area of the lawn
...area of the lawn	...formula for the area

The formula for the area of the lawn is:

$A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$



To calculate s, substitute: $a = 150$, $b = 200$, and $c = 300$

$s = \dfrac{150 + 200 + 300}{2}$, or 325

Calculate A. Substitute: $s = 325$, $a = 150$, $b = 200$, and $c = 300$

$A = \sqrt{325(325 - 150)(325 - 200)(325 - 300)}$

$A = \sqrt{325(175)(125)(25)}$

$\doteq 13\,331.71$

The area of the field is approximately 13 331.71 m^2.

Each bag of fertilizer covers an area of 900 m^2.

The number of bags needed to cover 13 331.71 m^2 is: $\dfrac{13\,331.71}{900} \doteq 14.8$

So, about 15 bags of fertilizer are needed.

The cost of the 15 bags of fertilizer is: 15(\$19.98) = \$299.70

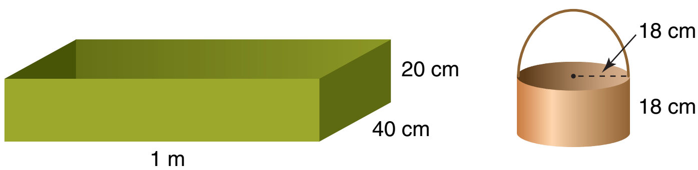
It costs \$299.70 to fertilize the lawn.

Example 3 Choosing Formulas and Converting Measures

Materials

- scientific calculator

A landscaper uses a bucket with radius 18 cm and height 18 cm to pour soil into a rectangular planter that measures 1 m by 40 cm by 20 cm.



How many buckets of soil are needed to fill the planter?

Solution

Convert all measurements to the same units.
1 m = 100 cm
Find the volume of soil each object can hold.

Planter
Use the formula for the volume of a rectangular prism: $V = lwh$ where V is the volume, l is the length, w is the width, and h is the height
Substitute: $l = 100$, $w = 40$, and $h = 20$

$$\begin{aligned} V &= (100)(40)(20) \\ &= 80\ 000 \end{aligned}$$

The planter holds 80 000 cm^3 of soil.

Bucket
Use the formula for the volume of a cylinder: $V = \pi r^2 h$ where V is the volume, r is the radius, and h is the height
Substitute: $r = 18$ and $h = 18$

$$\begin{aligned} V &= \pi(18)^2(18) \\ &\doteq 18\ 321.77 \end{aligned}$$

The bucket holds about 18 321.77 cm^3 of soil.
So, the number of buckets of soil needed is: $\dfrac{80\ 000}{18\ 321.77} \doteq 4.4$
About 4 buckets of soil are needed.

Problems in landscaping, construction, and design often involve the use of geometric formulas. The measurements substituted into these formulas must be in the same units.

Planning and organizing your solution

Organization is an important part of solving multi-step problems. The answers to these questions may be helpful in planning your solution.

- What formulas or relationships can be used?
- What numerical information is given?
- What numerical information do you need to find or estimate?
- What units of measurement are used?
  Do you need to convert from one set of units to another?

These problem-solving strategies may also be helpful.

- Work backward to determine what information you need.
- Work forward from the formulas and information you are given.
- Make a checklist of variables and their values.

Practice

A

■ For help with question 1, see Example 1.

1. The area, A, of a rectangle with length l and width w is $A = lw$. Find the area of a rectangle with each length and width.

a) $l = 10$ m, $w = 4$ m
b) $l = 6$ cm, $w = 8$ cm
c) $l = 9.5$ m, $w = 4.2$ m
d) $l = 8.4$ cm, $w = 7.2$ cm

2. The density, D, of an object with mass M and volume V is $D = \frac{M}{V}$. Determine the density of an object with each mass and volume.

a) $M = 200$ g, $V = 10$ cm^3
b) $M = 45$ g, $V = 7$ cm^3
c) $M = 7.8$ kg, $V = 2.6$ L
d) $M = 10$ kg, $V = 5.4$ L

3. The formula $S = 0.6T + 331.5$ gives the approximate speed of sound in air, S metres per second, when the air temperature is T degrees Celsius. Determine the speed of sound at each air temperature.

a) 30°C
b) −15°C
c) 10°C
d) −25°C

In Canada, temperatures are given in degrees Celsius, but in the United States, they are given in degrees Fahrenheit.

4. We can use the formula $C = \frac{5(F - 32)}{9}$ to convert degrees Fahrenheit, F, to degrees Celsius, C. Determine the Celsius equivalent of each Fahrenheit temperature.

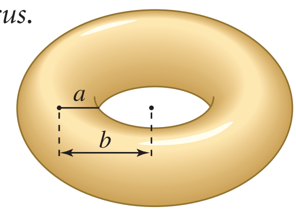
a) 77°F
b) 212°F
c) 50°F
d) −4°F

5. The approximate pressure, P kilopascals, exerted on the floor by the heel of a shoe is given by the formula $P = \frac{100m}{h^2}$, where m kilograms is the wearer's mass and h centimetres is the width of the heel. Determine the pressure exerted by the heel of each person's shoe.

	Person's mass (kg)	Shoe heel width (cm)
a)	80	6
b)	60	1.5
c)	55	3
d)	75	4.5

B

6. A doughnut and an inner tube are examples of a *torus*. The volume of a torus is given by the formula $V = 2\pi^2a^2b$. A dog chew toy is a torus with $a = 1$ cm and $b = 5$ cm. Determine the volume of rubber in the toy.



7. In a round-robin tournament, each team plays every other team once. The formula $G = \frac{n(n-1)}{2}$ gives the number of games G that must be scheduled for n teams.

a) How many games must be scheduled for 6 teams?

b) Will the number of games double if the number of teams doubles? Justify your answer.

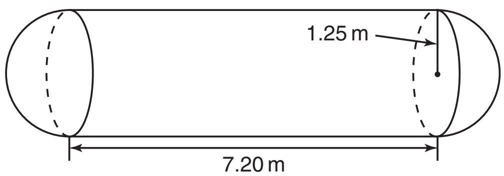
For help with question 8, see Example 2.

8. Vinh makes and sells T-shirts. The cost, C dollars, to produce n T-shirts is given by $C = 300 + 7n$. The revenue, R dollars, earned when n T-shirts are sold is given by $R = n\left(15 - \frac{n}{200}\right)$.

a) Determine the cost of making 200 T-shirts.

b) Profit is the difference between revenue and cost. Determine the profit from making and selling 1000 shirts.

9. A fuel storage tank consists of a cylinder with radius 1.25 m and length 7.20 m, with hemispheres of radius 1.25 m at each end.



a) Determine the surface area of the tank. Use the formula $SA = 4\pi r^2 + 2\pi rl$, where SA is the surface area of the tank, r is its radius, and l is the length of the cylinder.

b) Determine the cost to cover the tank with 2 coats of paint. One can of paint costs \$34.99 and covers an area of 29 m^2.

10. Body surface area is used to calculate drug dosages for cancer chemotherapy. The formula $B = \sqrt{\frac{w \times h}{3600}}$ gives the body surface area, B square metres, of an individual with height h centimetres and mass w kilograms.

a) Determine the body surface area of a child 102 cm tall with a mass of 21 kg.

b) The recommended child's dosage of a chemotherapy drug is 20 mg/m^2. How much medicine should the child in part a receive?

■ For help with question 11, see Example 3.

11. The bottle on an office water dispenser is a cylinder with radius 13.5 cm and height 49.1 cm. The paper cones from which people drink are 9.5 cm high with radius 3.5 cm. How many full cones of water can be dispensed?

12. A paving contractor has been hired to lay 6 cm of compacted asphalt on a road 12-m wide and 3.5-km long.
Each cubic metre of compacted asphalt has mass 2.5 t.
How many tonnes of asphalt should the contractor order?

13. **Assessment Focus** A hard rubber ball with radius 2.0 cm sells for $1.25.

a) Calculate the volume of the ball.

b) Suppose the radius is doubled. Does the volume double? Explain.

c) What would you charge for a ball with double the radius?
Justify your answer.

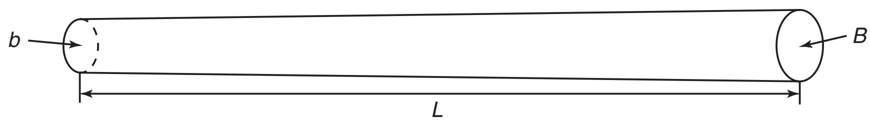
14. Samuel owns a pool maintenance company. One of his jobs is to chlorinate pool water. A single chlorine treatment requires 45 g of powdered chlorinator per 10 000 L of water. The chlorinator is sold in a 11.4-kg bucket that costs $54.99.

a) One of Samuel's clients has a swimming pool 18 m long and 10 m wide with an average depth of 2.5 m. How many litres of water does the pool hold? Explain.

b) How many grams of powdered chlorinator are required for a single treatment?

c) What is the cost of a single treatment? Explain.

15. In the forestry industry, it is important to estimate the volume of wood in a log. One formula that is used is $V = \frac{1}{2}L(B + b)$, where V cubic metres is the volume of wood, L metres is the length of the log, and B and b are the areas of the ends in square metres.



Estimate the volume of wood in a log with length 3.7 m and end diameters 30 cm and 40 cm.

16. **Literacy in Math** Create a matrix or checklist for the quantities in question 15. Write the given numerical values. Explain how you found the other values.

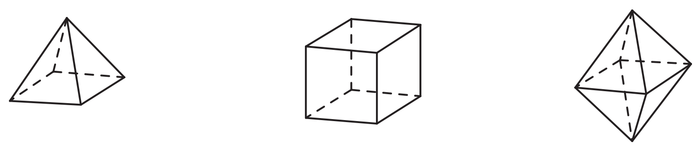
C **17.** *Example 1* introduced *Young's formula* for calculating a child's medicine dose, C milligrams: $C = \frac{Ag}{g + 12}$, where A represents the adult dose in milligrams and g represents the child's age, in years.

a) For a 6-year-old child, what fraction of the adult's dose is the child's dose? Explain how this fraction changes for older children.

b) For a given age, is the relationship between a child's dose and an adult's dose linear? Justify your answer.

18. *Euler's formula* relates the number of vertices (V), faces (F), and edges (E) of a polyhedron. Determine the value of $V + F - E$ for each polyhedron.

a) square pyramid **b)** cube **c)** octahedron



What do you think Euler's formula is? Explain.

In Your Own Words

Explain what is meant by this statement.
"The thinking and organizing you do to solve a multi-step problem is often backward from the presentation of the final solution."
Use an example to illustrate your explanation.

6.2 Rearranging Formulas

Travel agents make sure that their clients know what weather to expect at their destination. The formula $C = \frac{5(F-32)}{9}$ converts a temperature in degrees Fahrenheit, F, to degrees Celsius, C.

Investigate

Inverse Operations

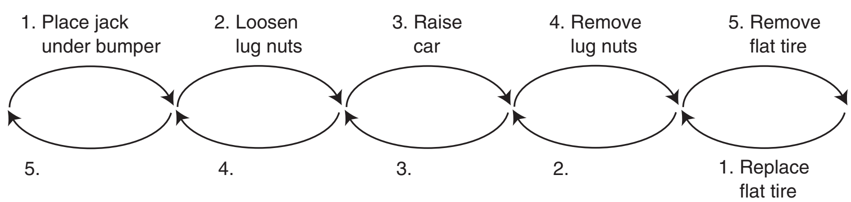
We can convert from Celsius to Fahrenheit by rearranging the formula $C = \frac{5(F-32)}{9}$ to isolate F.

One way to do this is to use inverse operations.
Inverse operations "undo" or reverse each other.

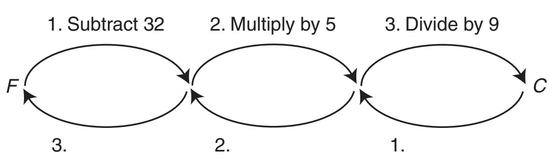
Work with a partner.
For each arrow diagram:

- Which operations will "undo" the sequence of operations in the top row of the diagram?
- Copy and complete the diagram.

Changing a flat tire



Converting between degrees Fahrenheit and degrees Celsius



Reflect

- How are the steps and operations in the top row of each arrow diagram related to the steps and operations in the bottom row of the diagram? Why are they related this way?
- List three different mathematical operations and their inverses.

Connect the Ideas

Rearranging formulas

Formulas usually express one variable in terms of one or more variables. We can use our knowledge of equations and inverse operations to rewrite the formula in terms of a different variable.

Example 1 Isolating a Variable

Rearrange each formula to isolate the indicated variable.

a) The amount, A dollars, of an investment is given by the formula $A = P + I$, where P dollars is the principal and I dollars is the interest earned. Isolate P.

b) The volume, V cubic metres, of a rectangular prism with length l metres, width w metres, and height h metres, is given by the formula $V = lwh$. Isolate h.

c) Ohm's Law, $I = \frac{V}{R}$, relates the current, I amperes, running along an electrical circuit to the voltage, V volts, and the resistance, R ohms. Isolate V.

Solution

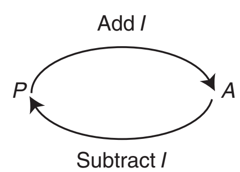
Use an arrow diagram to determine the inverse operations needed.

a) $A = P + I$

To isolate P, subtract I from each side.

$A - I = P + I - I$

$A - I = P$

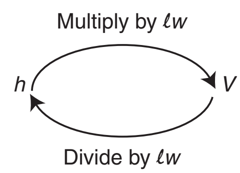


b) $V = lwh$

To isolate h, divide each side by lw.

$\frac{V}{lw} = \frac{lwh}{lw}$

$\frac{V}{lw} = h$

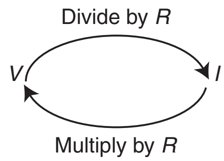


c) $I = \dfrac{V}{R}$

To isolate V, multiply each side by R.

$$I \times R = \frac{V}{R} \times R$$

$$IR = V$$

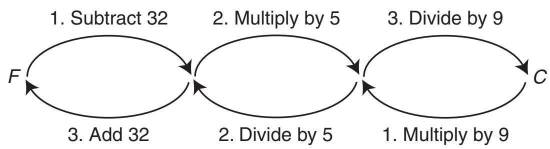


Example 2 Solving Problems by Rearranging a Formula

Convert 30°C to degrees Fahrenheit. Use the formula $C = \frac{5(F - 32)}{9}$.

Solution

Rearrange the formula to isolate F. Use an arrow diagram to determine the inverse operations required.



Method 1

Isolate F, then substitute.

$$C = \frac{5(F - 32)}{9}$$

Multiply each side by 9.

$$C \times 9 = \frac{5(F - 32)}{9} \times 9$$

$$9C = 5(F - 32)$$

Divide each side by 5.

$$\frac{9C}{5} = \frac{5(F - 32)}{5}$$

$$\frac{9C}{5} = F - 32$$

Add 32 to each side.

$$\frac{9C}{5} + 32 = F - 32 + 32$$

$$\frac{9C}{5} + 32 = F$$

Substitute: $C = 30$

$$F = \frac{9(30)}{5} + 32$$

$$F = 86$$

Method 2

Substitute, then solve for F.

$$C = \frac{5(F - 32)}{9}$$

Substitute: $C = 30$

$$30 = \frac{5(F - 32)}{9}$$

Multiply each side by 9.

$$30 \times 9 = \frac{5(F - 32)}{9} \times 9$$

$$270 = 5(F - 32)$$

Divide each side by 5.

$$\frac{270}{5} = \frac{5(F - 32)}{5}$$

$$54 = F - 32$$

Add 32 to each side.

$$54 + 32 = F - 32 + 32$$

$$86 = F$$

30°C is equivalent to 86°F.

Choosing a strategy

The "isolate, then substitute" and the "substitute, then solve" strategies produce the same result. Sometimes, one strategy is more efficient than the other.

- Isolate the variable first if you will have to calculate it several times.
- Substitute first if the numbers are simple or rearranging the formula is difficult.

Example 3 Solving Problems Involving Powers

Materials

- scientific calculator

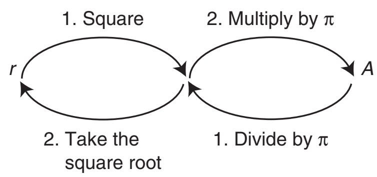
a) The area, A, of a circle with radius r is $A = \pi r^2$.
Use the formula $A = \pi r^2$ to determine the radius of a circular oil spill that covers an area of 5 km^2.

b) The volume, V, of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.
Use the formula $V = \frac{4}{3}\pi r^3$ to determine the radius of a Nerf ball with volume 1 m^3.

Solution

Powers and roots are inverse operations.
To "undo" squaring, take the square root.
To "undo" cubing, take the cube root.

a) Draw an arrow diagram.



In general, the inverse of the nth power is the nth root: $\sqrt[n]{}$
For example, $\sqrt[3]{64} = 4$ since $4^3 = 64$.

Substitute $A = 5$ in the formula $A = \pi r^2$.

$$5 = \pi r^2 \qquad \text{Divide each side by } \pi.$$

$$\frac{5}{\pi} = r^2 \qquad \text{Take the square root of each side.}$$

$$\sqrt{\frac{5}{\pi}} = r \qquad \text{Evaluate the left side.}$$

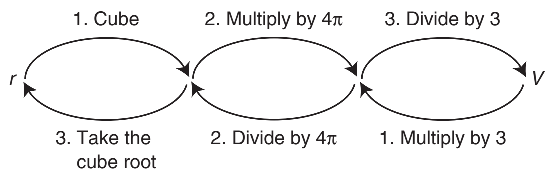
$$1.26 \doteq r$$

The radius of the oil spill is about 1.26 km.

In real-world situations, variables usually represent positive quantities, so take the positive root.

To evaluate $\sqrt{\frac{5}{\pi}}$ press:
2nd x^2 5 ÷ π) ENTER

b) Draw an arrow diagram.



To evaluate $\sqrt[3]{\frac{3}{4\pi}}$ press:

3 2nd ^ (3 ÷

(4 π)) ENTER

Substitute $V = 1$ in the formula $V = \frac{4}{3}\pi r^3$.

$1 = \frac{4}{3}\pi r^3$ Multiply each side by 3.

$3 = 4\pi r^3$ Divide each side by 4π.

$\frac{3}{4\pi} = r^3$ Take the cube root of each side.

$\sqrt[3]{\frac{3}{4\pi}} = r$ Evaluate the left side.

$0.62 \doteq r$

The radius of the Nerf ball is about 0.62 m.

Practice

For questions 1 to 4, use an arrow diagram to determine the inverse operations needed.

A

- For help with question 1, see Example 1.

1. The accounting formula $A = L + E$ relates assets A, liability L, and owners' equity E.

a) Isolate L. **b)** Isolate E.

2. The profit, P, earned by a business is given by the equation $P = R - C$, where R is the revenue and C is the cost.

a) Isolate R. **b)** Isolate C.

3. The area, A, of a parallelogram is given by the equation $A = bh$, where b is the length of the base and h is the height.

a) Isolate b. **b)** Isolate h.

4. The density, D, of an object is given by the equation $D = \frac{M}{V}$, where M is the object's mass and V is the object's volume.

a) Isolate M. **b)** Isolate V.

- For help with question 5, see Example 2.

5. A company uses the formula $a + s = 90$ to determine when an employee can retire with a full pension. In the formula, a is the employee's age and s is the number of years of service.

a) Solve for s when $a = 58$.

b) Solve for a when $s = 27$.

6. The formula $E = Rt$ gives the money earned, E dollars, for working at a rate of R dollars per hour for t hours. Jennie earns \$12 per hour. How many hours does she have to work to earn each amount?

a) \$420 **b)** \$126 **c)** \$504

B

7. Use the formula $E = Rt$ from question 6.

a) Drew works 35 h and earns \$542.50. What is his hourly rate of pay?

b) Did you substitute and solve or isolate and substitute? Explain.

8. The formula $S = 0.6T + 331.5$ gives the speed of sound in air, S metres per second, at an air temperature of T degrees Celsius.

a) Draw an arrow diagram to show the steps needed to isolate T in the formula.

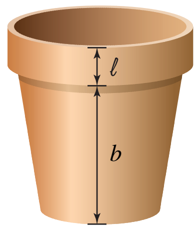
b) Isolate T.

c) Determine the air temperature for each speed of sound.

i) 343.5 m/s **ii)** 336 m/s **iii)** 328.5 m/s

9. A shoe store uses the formula $s = 3f - 21$ to model the relationship between a woman's shoe size, s, and her foot length, f, in inches. Nalini wears a size $7\frac{1}{2}$ shoe. Estimate her foot length to the nearest tenth of an inch.

10. The formula $H = nl + b$ gives the height, H, of n stacked containers, where each container has lip height l and base height b. Zoë is stacking flower pots with an 8-cm lip height and 50-cm base height at a garden centre. For safety reasons, the maximum allowable height of the stack is 1.3 m. How many pots can Zoë put in one stack? Justify your answer.



11. Office placement agencies use the formula $s = \frac{w - 10e}{m}$ to determine keyboarding speed. In the formula, s is the keyboarding speed in words per minute, w is the number of words typed, e is the number of errors made, and m is the number of minutes of typing.

a) Mark types 450 words in 5 min and makes 12 errors. What is his keyboarding speed?

b) If Rana makes no errors, how many words would she have to type in 5 min to have the same keyboarding speed as Mark?

12. Airplane pilots use the formula $s = \frac{d}{t}$ to estimate flight times. In the formula, s is the average speed, d is the distance travelled, and t is the flight time.

a) Estimate the flight time from Ottawa to Thunder Bay, a distance of 1100 km. Assume that the airplane flies at an average speed of 350 km/h.

b) Describe the operations you used to isolate t.

13. In house construction, the safe load, m kilograms, that can be supported by a beam with length l metres, thickness t centimetres, and height h centimetres is given by the formula $m = \frac{4th^2}{l}$.

a) Determine t when $m = 500$ kg, $l = 4$ m, and $h = 10$ cm.

b) Determine l when $m = 250$ kg, $t = 10$ cm and $h = 5$ cm.

c) How are the steps used to solve for a variable in the denominator of a fraction similar to the steps used to solve for a variable in the numerator? How are they different?

■ For help with question 14, see Example 3.

14. The equation $V = \frac{1}{6}\pi d^3$ gives the volume, V, of a sphere in terms of its diameter, d. Use the formula to determine the diameter of a ball with volume 1000 cm^3.

1 kW = 1000 W

15. The formula $P = \frac{r^2 s^3}{2}$ gives the approximate power, P watts, generated by a wind turbine with radius r metres in a wind of speed s metres per second. The Exhibition Place Wind Turbine in Toronto has radius about 24 m. Determine the wind speed when the turbine generates 500 kW of power.

16. **Assessment Focus** The volume of a cylinder is given by the formula $V = \pi r^2 h$, where V is the volume, r is the radius, and h is the height.

a) Rearrange the formula to isolate r.
Explain your choice of inverse operations.

b) Determine the radius of a cylindrical fuel tank that is 16 m high and holds 200 m^3 of fuel.

c) Determine the height of a cylindrical mailing tube with volume 2350 cm^3 and radius 5 cm. Justify your choice of strategy.

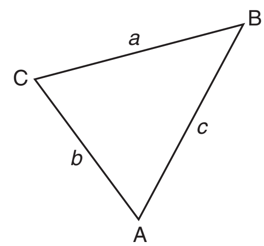
17. **Literacy in Math** Use a graphic organizer to summarize the pairs of inverse operations that can be used to rearrange a formula. Explain the reason for your choice of organizer.

C **18.** A police officer uses the formula $S = 15.9\sqrt{df}$ to estimate the speed of a vehicle when it crashed. In the formula, S kilometres per hour is the speed of the vehicle, d metres is the length of the skid marks left on the road, and f is the coefficient of friction, a measure of the traction between the road surface and the vehicle's tires.

a) The skid marks left on a dry road measure 40 m.
What was the speed of the vehicle if $f \doteq 0.85$ for a dry road?

b) A car travelling at 30 km/h skids and crashes in an icy parking lot.
Estimate the length of the skid marks at the crash site if $f \doteq 0.35$ for an icy road.

19. In Chapter 1, you used the Cosine Law $a^2 = b^2 + c^2 - 2bc \cos A$ to solve oblique triangles.



a) Rearrange the formula to isolate cos A.

b) Determine the measure of the greatest angle in a triangle with side lengths 5 m, 6 m, and 7 m.

c) Why did we rearrange the formula for cos A instead of $\angle A$?

In Your Own Words

Choose a reversible routine from daily life such as setting the table or getting dressed. Explain why reversing the routine means undoing each step in the opposite order. Explain how this idea is used to rearrange a formula. Include an example in your explanation.

6.3 Laws of Exponents

Many formulas in science, business, and industry involve integer exponents. For example, the formula $V = 0.05hc^2$ is used in the forestry industry to estimate the volume of wood in a tree. In the formula, V is the volume of wood in the tree, h is the height of the tree, and c is the circumference of the trunk.

Investigate — Simplifying Products and Quotients of Powers

Materials
- TI-89 calculator (optional)

Part A: Expanding Products and Quotients of Powers

- Copy and complete each table.
- Describe the relationship between the exponents in the original expression and the exponent in the expression as a single power.

Multiplying powers

Original expression	Powers in expanded form	Expression as a single power
$b^2 \times b^4$		
$b^5 \times b^{-3}$	$(\overset{1}{\cancel{b}} \times \overset{1}{\cancel{b}} \times \overset{1}{\cancel{b}} \times b \times b)\left(\dfrac{1}{\underset{1}{\cancel{b}} \times \underset{1}{\cancel{b}} \times \underset{1}{\cancel{b}}}\right)$	b^2
$b^{-8} \times b^5$		
$b^4 \times b^2 \times b^{-1}$		

Power of a power

Original expression	Powers in expanded form	Expression as a single power
$(b^2)^3$		
$(b^4)^3$		
$(b^{-3})^5$	$b^{-3} \times b^{-3} \times b^{-3} \times b^{-3} \times b^{-3}$	b^{-15}
$(b^4)^{-1}$		

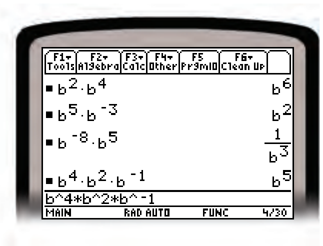
Dividing powers

Original expression	Powers in expanded form	Expression as a single power
$\frac{b^7}{b^2}$		
$b^5 \div b^2$		
$\frac{b^2}{b^3}$	$\frac{\overset{1}{\cancel{b}} \times \overset{1}{\cancel{b}}}{\underset{1}{\cancel{b}} \times \underset{1}{\cancel{b}} \times b} = \frac{1}{b}$	b^{-1}
$b^3 \div b^7$		

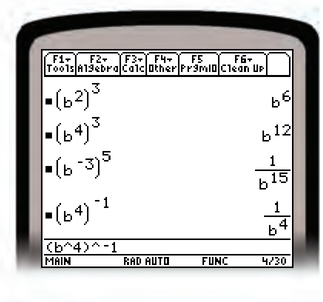
Part B: Using a CAS

The expressions in the tables in Part A were entered in a computer algebra system (CAS). These results were obtained.

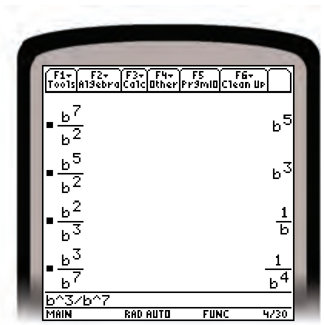
Multiplying powers



Power of a power



Dividing powers



- Compare your answers in Part A with those from the CAS. Explain any differences in the answers.
- Complete these exponent laws.

  Multiplying powers: $a^m \times a^n = a^?$

  Power of a power: $(a^m)^n = a^?$

  Dividing powers: $a^m \div a^n = a^?$

Reflect

- How does the CAS display powers with negative exponents? Why do you think it displays them that way?
- Suppose you forget the exponent laws or are not sure how to apply them. What strategies can you use to help?

Connect the Ideas

Definitions of integer exponents

a^{-n} is the reciprocal of a^n.

The definition of a power depends on whether the exponent is a positive integer, zero, or a negative integer.

- Positive integer exponent $\quad a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ factors}}$
- Zero exponent $\quad a^0 = 1, a \neq 0$
- Negative integer exponent $\quad a^{-n} = \frac{1}{a^n}, a \neq 0$

Laws of exponents

The definitions of integer exponents lead to general rules for working with exponents.

The exponent laws apply to numerical and variable bases. When the base is a variable, we assume that it is not 0.

Laws of exponents

- Multiplication law $\quad a^m \times a^n = a^{m+n}$
- Division law $\quad a^m \div a^n = a^{m-n}, a \neq 0$
- Power of a power law $\quad (a^m)^n = a^{mn}$

m and n are any integer.

The laws can be used to evaluate numerical expressions and to simplify algebraic expressions.

Example 1 Applying the Laws of Exponents

Simplify. Evaluate where possible.

a) $5^4 \times 5^{-2}$

b) $\dfrac{(-6)^2}{(-6)^{-1}}$

c) $(m^5)^{-3}$

d) $\dfrac{a^2a^{-5}}{(a^{-2})^3}$

a^2a^{-5} means $a^2 \times a^{-5}$.

Solution

a)
$$\begin{aligned} 5^4 \times 5^{-2} &= 5^{4+(-2)} \\ &= 5^2 \\ &= 25 \end{aligned}$$

b)
$$\begin{aligned} \frac{(-6)^2}{(-6)^{-1}} &= (-6)^{2-(-1)} \\ &= (-6)^{2+1} \\ &= (-6)^3 \\ &= -216 \end{aligned}$$

c)
$$\begin{aligned} (m^5)^{-3} &= m^{5 \times (-3)} \\ &= m^{-15} \\ &= \frac{1}{m^{15}} \end{aligned}$$

By convention, a simplified algebraic expression contains only positive exponents. So, we write m^{-15} as $\frac{1}{m^{15}}$.

d)
$$\begin{aligned} \frac{a^2a^{-5}}{(a^{-2})^3} &= \frac{a^{2+(-5)}}{a^{-2 \times 3}} \\ &= \frac{a^{-3}}{a^{-6}} \\ &= a^{-3-(-6)} \\ &= a^3 \end{aligned}$$

Example 2 Using Exponents in an Application

Materials

- scientific calculator

The number of hybrid vehicles sold in the United States, S, can be modelled by the formula $S = 199\ 148(2.39)^n$, where n is the number of years since 2005.

This article is an excerpt of a *CBS News* article from May 4, 2006.

Hybrid Vehicle Sales More than Double
Registrations in the United States for new hybrid vehicles rose to 199 148 in 2005, a 139% increase from the year before...

The model assumes that hybrid car sales will continue to grow at the rate of increase given in the article.

a) Evaluate $S = 199\ 148(2.39)^n$ when $n = 0$. What does the answer represent?

b) Estimate the number of hybrid vehicles sold in 2004.

c) Predict the number of hybrid vehicles that will be sold in 2007.

Solution

a) $S = 199\ 148(2.39)^0$
$= 199\ 148$
This represents the number of vehicles sold in 2005.

$(2.39)^0 = 1$

A zero exponent represents an initial value. Positive exponents represent going forward in time. Negative exponents represent going back in time.

b) Substitute $n = -1$ in $S = 199\ 148(2.39)^n$.
$S = 199\ 148(2.39)^{-1}$
$\doteq 83\ 326$
About 83 000 hybrids were sold in 2004.

2004 is 1 year before 2005.

c) Substitute $n = 2$ in $S = 199\ 148(2.39)^n$.
$S = 199\ 148(2.39)^2$
$\doteq 1\ 137\ 553$
If the rate of growth given in the article continues, more than 1 million hybrids will be sold in 2007.

2007 is 2 years after 2005.

Example 3 Simplifying Expressions

Evaluate each expression for $a = 1$, $b = -2$, and $c = 3$.

a) $(a^{-2}b)(a^3b^4)$

b) $\dfrac{a^{-4}b^5c^2}{ab^3c}$

c) $(a^5b^2)^3$

d) $(2a^2b)^5$

Solution

Simplify first, then evaluate.

a)
$$\begin{aligned}(a^{-2}b)(a^3b^4) &= a^{-2+3}b^{1+4}\\ &= a^1b^5\\ &= (1^1)(-2)^5\\ &= -32\end{aligned}$$

b)
$$\begin{aligned}\frac{a^{-4}b^5c^2}{ab^3c} &= a^{-4-1}b^{5-3}c^{2-1}\\ &= a^{-5}b^2c^1\\ &= \frac{b^2c}{a^5}\\ &= \frac{(-2)^2(3)}{(1)^5}\\ &= 12\end{aligned}$$

$c = c^1$

c)
$$\begin{aligned}(a^5b^2)^3 &= a^{5\times 3}b^{2\times 3}\\ &= a^{15}b^6\\ &= (1)^{15}(-2)^6\\ &= 64\end{aligned}$$

d)
$$\begin{aligned}(2a^2b)^5 &= 2^{1\times 5}a^{2\times 5}b^{1\times 5}\\ &= 2^5a^{10}b^5\\ &= 32(1)^{10}(-2)^5\\ &= -1024\end{aligned}$$

Practice

A

■ For help with questions 1 to 3, see Example 1.

A simplified expression contains only positive exponents.

1. Simplify, but do not evaluate.

a) $2^3 \times 2^4$ **b)** $3^1 \times 3^{-4}$ **c)** $(1.05)^{-3} \times (1.05)^4$

d) c^5c^4 **e)** $\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^5$ **f)** $a^4\,a^{-2}a$

2. Simplify, but do not evaluate.

a) $4^5 \div 4^2$ **b)** $\dfrac{5^3}{5^7}$ **c)** $(1.02)^{13} \div (1.02)^{10}$

d) $\dfrac{d^5}{d}$ **e)** $(-3)^7 \div (-3)^{-4}$ **f)** $\dfrac{h^{30}}{h^{20}}$

3. Simplify, but do not evaluate.

a) $(5^3)^2$ **b)** $(3^{-2})^5$ **c)** $[(-2)^{-4}]^3$

d) $(m^5)^4$ **e)** $(r^{-10})^{-2}$ **f)** $(a^3)^3$

4. Evaluate without a calculator.

a) 10^4 b) 9^0 c) 3^{-2}

d) 2^{-3} e) $\left(\frac{2}{3}\right)^2$ f) $\left(\frac{1}{5}\right)^{-2}$

5. Evaluate.

a) 3^9 b) 4^{-2} c) $(-4)^{-2}$ d) -2^4

e) 0.5^{-2} f) $\left(\frac{2}{5}\right)^3$ g) 1.05^{27} h) $(-1)^{55}$

Which expressions could you evaluate without a calculator? Explain.

6. Simplify each expression.
Which exponent laws did you use?

a) d^5d^{-2} b) $(x^{-5})^2$ c) $\frac{c^{11}}{c^{-3}}$ d) $\left(\frac{1}{z^3}\right)^{-6}$

e) $n^4n^{-13}n^7$ f) $w^{-8}(w^3)^2$ g) $\frac{s^5s^4}{s^{-3}}$ h) $\frac{(t^4)^{-5}}{t^6}$

B

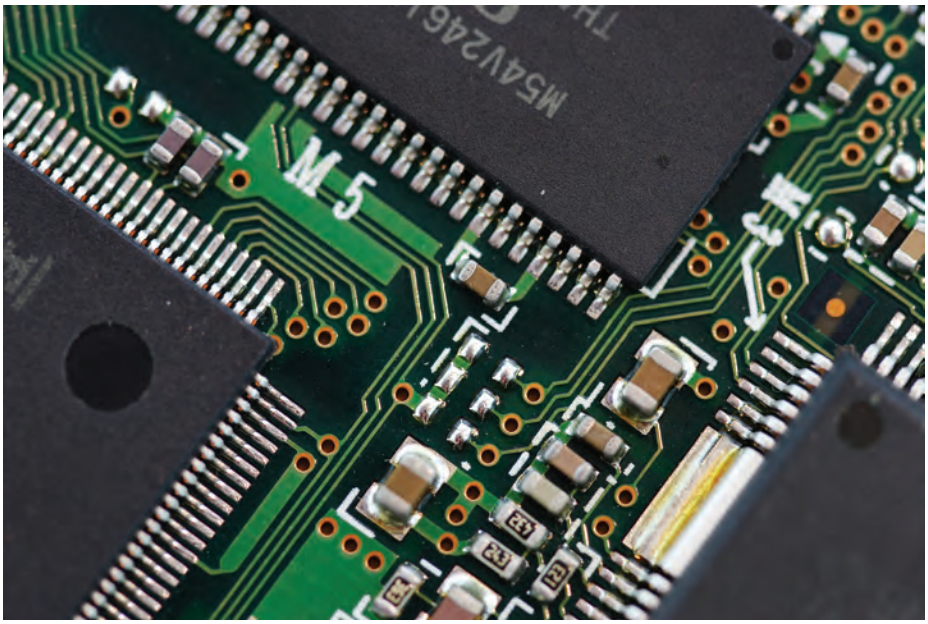
For help with questions 7 and 8, see Example 2.

7. Evaluate $N = 400(2)^n$ for each value of n.

a) $n = 3$ b) $n = 0$ c) $n = -3$

8. Computer power has been doubling approximately every 2 years as more and smaller transistors have been integrated to build better computer chips. The number of transistors, T, in a chip has increased according to $T = 4500\ (1.4)^n$, where n is the number of years since 1974. Determine the number of transistors in a computer chip in each year.

a) 1974 b) 1972 c) 2002



For help with questions 9 to 11, see Example 3.

9. Evaluate for $x = 2$ and $y = -3$ without a calculator.

a) x^{-4}

b) 5^y

c) x^y

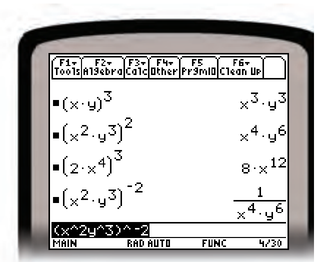
d) y^x

10. a) Substitute $x = 2$ in the expression $\dfrac{x^5x^4}{(x^2)^3}$. Evaluate without simplifying.

b) Simplify $\dfrac{x^5x^4}{(x^2)^3}$, and then evaluate at $x = 2$.

c) Compare the methods in parts a and b. Describe the advantages of each method.

11. Use the CAS calculator screen below.



a) How are the exponents of the original expressions related to the exponents of the simplified expressions?

b) Explain the relationship by writing the original expressions in expanded form and simplifying.

c) Complete the law that generalizes the pattern: $(a \times b)^n = a^?b^?$

d) Simplify.

i) $(2f)^4$

ii) $(a^3b)^4$

iii) $(s^{-3}v^4)^5$

iv) $(5h)^{-2}$

12. Evaluate for $x = 2$, $y = -3$, and $z = 5$.

a) $x^2y^4x^3y^{-2}$

b) $\dfrac{x^3y^3z}{xy^4z^{-2}}$

c) $\dfrac{(5x)^2(2y)^3}{10xy^2}$

d) $(xyz)^4x^{-5}y^7z^{-5}$

13. **Assessment Focus**

a) Evaluate $\dfrac{a^2b^5c^5}{ab^{-3}c^4}$ for $a = 6$, $b = 2$, and $c = -10$. Explain your method.

b) Terry rewrote $(5r)^3$ as $5r^3$ and $5r^{-2}$ as $\dfrac{1}{5r^2}$ on a test. Explain the mistakes Terry made. What strategies might Terry use to help him avoid making these mistakes in the future?

14. The formula $V = \pi r^2 h$ gives the volume, V, of a cylinder with radius r and height h.

a) Determine the volume of a cylindrical gift tube with radius $2x$ and height $7x$.

b) Calculate the volume of the gift tubes when $x = 5$ cm and $x = 12$ cm.

15. **Literacy in Math** An excerpt of a *CBS News* article from May 4, 2006 is shown at the right.

a) Explain the phrase "has grown exponentially."

b) What quantities can you calculate from the given information?

c) Explain why the numbers in the second sentence are reasonably consistent with each other.

d) In *Example 2*, the 2004 sales estimate was 83 326 hybrids. Is this inconsistent with the estimate given in the article? Explain.

Hybrid Vehicle Sales More than Double
Hybrids accounted for 1.2 percent of the 16.99 million vehicles sold in the United States last year. In 2004, the 83 153 hybrids sold were 0.5 percent of the 16.91 million vehicles sold. The U.S. hybrid market has grown exponentially since 2000, when 7781 were sold.

C

16. Refer to *Example 2* and question 15.

a) Show that hybrid sales did not increase by 139% each year between 2000 and 2004.

b) Estimate the actual average growth rate between 2000 and 2004. Explain your method.

17. The formula $A = P(1 + i)^n$ gives the amount, A dollars, of a compound interest investment. In the formula, P dollars is the principal invested, i is the annual interest rate as a decimal, and n is the number of years.

a) Rearrange the formula to isolate P.

b) Rewrite the formula in part a using a negative exponent.

c) Evaluate the formulas in parts a and b for P when $A = \$1000$, $i = 6\%$, and $n = 5$ years. Which formula did you find easier to evaluate? Explain.

In Your Own Words

What are some mistakes you have made when working with exponents?
Why do you think you made these mistakes?
How might you avoid making these mistakes in the future?
Include examples in your explanation.

6.4 Patterns in Exponents

Coffee, tea, cola, and chocolate each contain caffeine. The formula $P = 100(0.87)^n$ models the percent, P, of caffeine left in your body n hours after you drink a caffeinated beverage. After half an hour, the percent of caffeine remaining in your body is given by the equation $P = 100(0.87)^{\frac{1}{2}}$.

Inquire

Exploring Rational Exponents

Materials

- TI-83 or TI-84 graphing calculator

Work with a partner.

Part A: Exploring the Meaning of $a^{\frac{1}{n}}$

1. The expressions in the table use the exponents 2, -2, and $\frac{1}{2}$.

$1^2 = 1$	$1^{-2} = 1$	$1^{\frac{1}{2}} = 1$
$2^2 = 4$	$2^{-2} = \frac{1}{4}$	$4^{\frac{1}{2}} = 2$
$3^2 = 9$	$3^{-2} = \frac{1}{9}$	$9^{\frac{1}{2}} = 3$
$4^2 = 16$	$4^{-2} = \frac{1}{16}$	$16^{\frac{1}{2}} = 4$

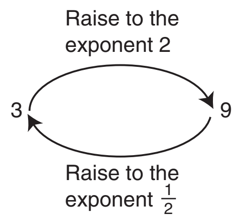
a) Determine the next 3 rows in the table. Explain your reasoning.

b) Compare the numbers in the first and second columns. Describe any relationships you see. What does it mean to raise a number to the exponent 2? To the exponent -2?

c) Think of any number, a. What can you say about the value of $a^{\frac{1}{2}}$?

An exponent that can be written as a fraction of integers is a **rational exponent**.

d) Compare the numbers in the first and third columns. Notice that the exponent $\frac{1}{2}$ appears to "undo," or reverse, the exponent 2.



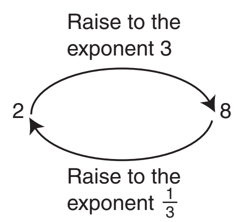
What do you think it means to raise a number to the exponent $\frac{1}{2}$? Confirm your answer by trying other examples on your calculator.

2. The patterns in the table use the exponents 3, −3, and $\frac{1}{3}$.

$1^3 = 1$	$1^{-3} = 1$	$1^{\frac{1}{3}} = 1$
$2^3 = 8$	$2^{-3} = \frac{1}{8}$	$8^{\frac{1}{3}} = 2$
$3^3 = 27$	$3^{-3} = \frac{1}{27}$	$27^{\frac{1}{3}} = 3$
$4^3 = 64$	$4^{-3} = \frac{1}{64}$	$64^{\frac{1}{3}} = 4$

a) Complete the next three lines in each pattern. Explain your reasoning.

b) Compare the numbers in the first and second columns. Describe any relationships you see. What does it mean to raise a number to the exponent 3? To the exponent −3?



c) Compare the numbers in the first and third columns. Notice that the exponent $\frac{1}{3}$ appears to "undo" or reverse the exponent 3. What do you think it means to raise a number to the exponent $\frac{1}{3}$? Confirm your answer by trying other examples on your calculator.

3. a) You have explored the meaning of $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$.

What do you think $a^{\frac{1}{4}}$ and $a^{\frac{1}{5}}$ mean?
Use a calculator to test your predictions.

b) How would you define $a^{\frac{1}{n}}$? Explain your reasoning.

a	$a^{\frac{1}{2}}$	$a^{\frac{3}{2}}$	$a^{\frac{5}{2}}$
1	1		
4	2		
9	3		
16	4		

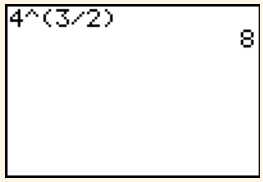
Brackets are needed around the exponent so that the calculator evaluates $4^{\frac{3}{2}}$, not $4^3 \div 2$.

Part B: Exploring the Meaning of $a^{\frac{m}{n}}$

4. **a)** Copy the table.

b)

Use your graphing calculator to complete the third column of the table. For example, to determine $4^{\frac{3}{2}}$, press:
4 [^] [(] 3 [÷] 2 [)] [ENTER]



c) Compare the numbers in corresponding rows of the second and third columns of the table.
How do the values of $a^{\frac{3}{2}}$ appear to be related to the values of $a^{\frac{1}{2}}$? Explain.

d) We can think of the exponent $\frac{3}{2}$ as the product $\frac{1}{2} \times 3$.
Explain why this allows us to rewrite $4^{\frac{3}{2}}$ as $(4^{\frac{1}{2}})^3$.
Evaluate each expression to show that they produce the same result.
How does this explain the relationship in part c?

e) How do you think the values of $a^{\frac{5}{2}}$ will be related to the values of $a^{\frac{1}{2}}$? Explain your reasoning. Use a calculator to complete the fourth column of the table. Were you correct? Explain.

a	$a^{\frac{1}{3}}$	$a^{\frac{2}{3}}$	$a^{\frac{5}{3}}$
1	1		
8	2		
27	3		
64	4		

5. Copy the table.

a) What do you think $a^{\frac{2}{3}}$ and $a^{\frac{5}{3}}$ mean? Explain your reasoning.

b) How do you think the values of $a^{\frac{2}{3}}$ and $a^{\frac{5}{3}}$ will be related to the value of $a^{\frac{1}{3}}$? Justify your answer.

c) Use a calculator to complete the table.
Were you correct in part b? Explain.

6. Use the results of questions 4 and 5.
How do you think $a^{\frac{m}{n}}$ is defined?
Explain your reasoning.

Practice

A

1. Explain the meaning of the exponent in each expression.

a) 8^3 b) 8^{-3} c) $8^{\frac{1}{3}}$ d) $8^{\frac{2}{3}}$

2. Evaluate each expression without using a calculator.

a) $9^{\frac{1}{2}}$ b) $49^{\frac{1}{2}}$ c) $64^{\frac{1}{2}}$

d) $27^{\frac{1}{3}}$ e) $(-8)^{\frac{1}{3}}$ f) $1000^{\frac{1}{3}}$

How do you know your answers are correct?

3. a) Evaluate.

i) $25^{\frac{1}{2}}$ ii) $25^{\frac{2}{2}}$ iii) $25^{\frac{3}{2}}$ iv) $25^{\frac{4}{2}}$ v) $25^{\frac{5}{2}}$

b) What pattern do you notice in the answers? Explain.

c) Write, then evaluate the next 3 terms in the pattern. Justify your answers.

4. a) Explain why $100^{\frac{1}{2}} = 10$.

b) How will the values of $100^{\frac{3}{2}}$, $100^{\frac{5}{2}}$, and $100^{\frac{7}{2}}$ be related to the value of $100^{\frac{1}{2}}$?

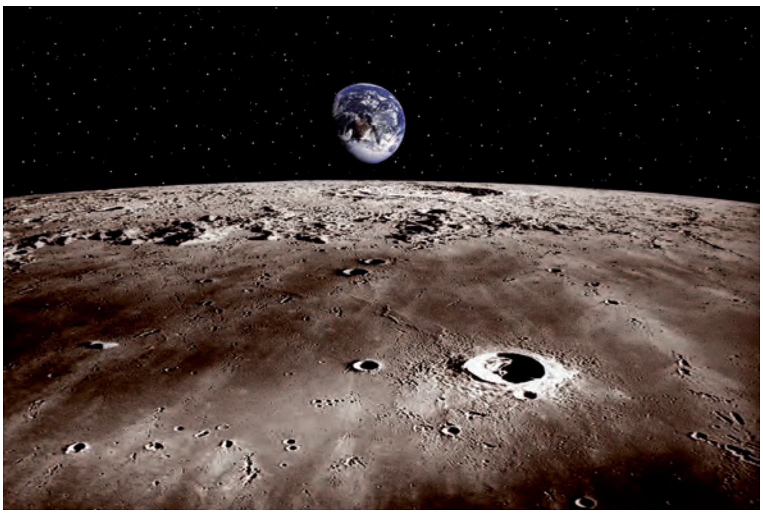
c) Use a calculator to determine the value of $100^{\frac{3}{2}}$, $100^{\frac{5}{2}}$, and $100^{\frac{7}{2}}$.
Were you correct in part b? Explain.

5. Rewrite using radicals and evaluate without a calculator.

a) $32^{\frac{1}{5}}$ b) $81^{\frac{1}{4}}$ c) $16^{\frac{3}{2}}$ d) $9^{\frac{5}{2}}$

e) $100^{\frac{3}{2}}$ f) $16^{\frac{3}{4}}$ g) $8^{\frac{4}{3}}$ h) $27^{\frac{3}{3}}$

How do you know that your answers are correct?



6. Use the table of values and graph of $y = 2^x$ shown here.

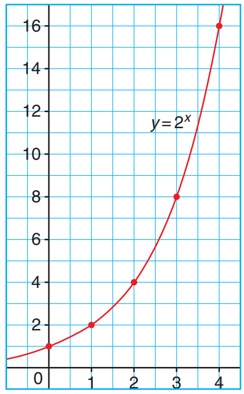
a) Explain why the value of $2^{\frac{1}{2}}$ must be between 1 and 2.

b) Use the graph to estimate the value of $2^{\frac{1}{2}}$ to the nearest tenth.

c) Which two whole numbers is $2^{\frac{3}{2}}$ between? Repeat for $2^{\frac{5}{2}}$ and $2^{\frac{7}{2}}$.

d) Estimate the value of $2^{\frac{3}{2}}$, $2^{\frac{5}{2}}$, and $2^{\frac{7}{2}}$ to the nearest tenth.

x	$y = 2^x$
0	1
1	2
2	4
3	8
4	16



7. The equation $P = 100(0.87)^x$ models the percent, P, of caffeine left in your body x hours after you consume it. Determine the value of P after each time.

a) $\frac{1}{2}$ h **b)** $\frac{3}{2}$ h **c)** 40 min

How do you know your answers are reasonable?

Reflect

- In the power, $x^{\frac{m}{n}}$, what does the numerator represent? What does the denominator represent? Explain.
- What steps do you take to evaluate the power $x^{\frac{m}{n}}$? Use examples in your explanation.

Mid-Chapter Review

6.1

1. The area, A, of a diamond shape with diagonal lengths d and D is $A = \frac{1}{2}dD$. Find the area of a diamond with each of these diagonal lengths.

a) $d = 4$ m, $D = 3$ m

b) $d = 47$ cm, $D = 68$ cm

2. During aerobic exercise, the maximum desirable heart rate, h beats per minute, is given by the formula $h = 198 - 0.9a$, where a is the person's age in years. Determine your maximum desirable heart rate.

3. Zan is planning to waterproof a rectangular driveway that is 12 m long and 5.5 m wide.

a) What is the area of the driveway?

b) One can of waterproofing sealer costs \$15.99 and covers an area of 30 m^2. How much will it cost Zan to waterproof the driveway?

4. The area, A, of a triangle with side lengths a, b, and c is given by the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$. The sides of a triangular plot of land measure 500 m, 750 m, and 1050 m. Land is priced at \$5400 per hectare (1 ha $=$ 10 000 m^2). Determine the value of the plot.

6.2

5. a) Describe the steps to rearrange the equation $y = 3x + 5$ to isolate x. Use an arrow diagram to determine the inverse operations needed.

b) Isolate x.

6. A car accelerates away from a stop sign. The formula $d = \frac{1}{2}at^2$ gives the distance, d metres, that the car travels in t seconds at an acceleration of a metres per second squared.

a) Find d when $a = 2$ m/s^2 and $t = 15$ s.

b) Find a when $d = 100$ m and $t = 5$ s.

c) Find t when $a = 0.01$ m/s^2 and $d = 20.48$ m.

7. The formula $I = Prt$ gives the simple interest, I dollars, earned on a principal of P dollars invested at an annual interest rate of r percent for t years. An investment of \$1000 earns \$131.25 interest in 2.5 years. What annual rate of interest was paid?

6.3

8. Evaluate without a calculator.

a) $(-3)^2$ b) -3^2 c) -3^{-2} d) $(-3)^{-2}$

9. Simplify.

a) $p^4 \times p^{-2}$ b) $p^3 \div p^8$ c) $(p^{-2})^5$

10. Simplify and evaluate for $x = -3$, $y = 4$, and $z = 5$.

a) $x^7\, y^{-2}\, x^3$ b) $\dfrac{x^5y^2z^3}{x^{-1}y^0z}$ c) $(2x^3)^2$

11. a) Write the next three terms in the pattern. Describe the pattern.

$4^0 \quad 4^{\frac{1}{2}} \quad 4^1 \quad 4^{\frac{3}{2}} \quad 4^2$

b) Evaluate each power in the pattern as a whole number or a fraction. Describe the pattern in the answers.

12. Evaluate without a calculator.

a) $16^{\frac{1}{2}}$ b) $64^{\frac{1}{3}}$ c) $25^{\frac{3}{2}}$

6.5 Rational Exponents

Many formulas in biology involve rational exponents. The formula $v = 0.783\left(\frac{s^{10}}{h^7}\right)^{\frac{1}{6}}$ approximately relates an animal's speed, v metres per second, to its stride length, s metres, and its hip height, h metres.

Investigate

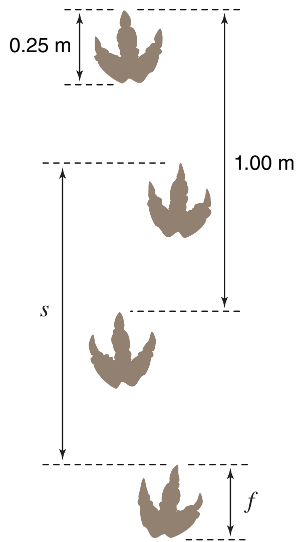
Calculating the Speed of a Dinosaur

Materials

- scientific calculator

Paleontologists use measurements from fossilized dinosaur tracks to estimate the speed at which the dinosaur travelled.

- The stride length, s, of a dinosaur is the distance between successive footprints of the same foot.
- The hip height, h, of a dinosaur is about 4 times the foot length, f.



Work with a partner.

Use the measurements on the diagram.

- Estimate the speed of the dinosaur.
  Use the formula $v = 0.783\left(\frac{s^{10}}{h^7}\right)^{\frac{1}{6}}$.

Reflect

- Describe your strategy. Explain how well your strategy worked.
- Compare your strategy with another pair's strategy.
  How are they the same? How are they different?

Connect the Ideas

Definition of $a^{\frac{1}{n}}$

You explored the meaning of rational exponents in Lesson 6.4. Mathematicians chose these meanings by extending the exponent laws to rational exponents.

Extending the exponent law $(a^m)^n = a^{mn}$ to include rational exponents gives:

$$\begin{aligned}\left(6^{\frac{1}{2}}\right)^2 &= 6^{\frac{1}{2}\times 2}\\ &= 6^1\\ &= 6\end{aligned}$$

$$\begin{aligned}\left(-6^{\frac{1}{2}}\right)^2 &= \left[(-1)^1 \times 6^{\frac{1}{2}}\right]^2\\ &= (-1)^{1\times 2} \times 6^{\frac{1}{2}\times 2}\\ &= (-1)^2 \times 6^1\\ &= 6\end{aligned}$$

$\sqrt{n}$ is the positive root of n, so $\sqrt{36} = 6$.

But: $(\sqrt{6})^2 = 6$ and $(-\sqrt{6})^2 = 6$

So, mathematicians defined: $6^{\frac{1}{2}} = \sqrt{6}$ and $-6^{\frac{1}{2}} = -\sqrt{6}$.

They also defined $6^{\frac{1}{3}} = \sqrt[3]{6}$, $6^{\frac{1}{4}} = \sqrt[4]{6}$, and so on.

A natural number is any number in the set 1, 2, 3, ...

Definition of $a^{\frac{1}{n}}$

$a^{\frac{1}{n}}$ is the nth root of a. That is, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

n is a natural number.

$a \geq 0$ if n is even.

Definition of $a^{\frac{m}{n}}$

The expression $a^{\frac{m}{n}}$ can be interpreted in two ways.

- $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m$

  Take the nth root of a, then raise the result to the exponent m.

  For example, $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = (2)^2 = 4$

- $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

Raise a to the exponent m, then take the nth root.

For example, $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

Definition of $a^{\frac{m}{n}}$

$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

m is an integer.

n is a natural number.

$a \geq 0$ if n is even.

Example 1 Evaluating Powers with Rational Exponents

Evaluate without a calculator.

a) $49^{\frac{1}{2}}$ **b)** $(-64)^{\frac{1}{3}}$ **c)** $32^{\frac{4}{5}}$ **d)** $0.04^{\frac{3}{2}}$

Solution

Rewrite each expression in radical form.

a) $$\begin{aligned} 49^{\frac{1}{2}} &= \sqrt{49} \\ &= 7 \text{ since } 7^2 = 49 \end{aligned}$$

b) $$\begin{aligned} (-64)^{\frac{1}{3}} &= \sqrt[3]{-64} \\ &= -4 \text{ since } (-4)^3 = -64 \end{aligned}$$

c) $$\begin{aligned} 32^{\frac{4}{5}} &= (32^{\frac{1}{5}})^4 \\ &= (\sqrt[5]{32})^4 \\ &= 2^4 \\ &= 16 \end{aligned}$$

d) $$\begin{aligned} 0.04^{\frac{3}{2}} &= (0.04^{\frac{1}{2}})^3 \\ &= (\sqrt{0.04})^3 \\ &= 0.2^3 \\ &= 0.008 \end{aligned}$$

Using rational exponents to solve equations

Rational exponents are useful for solving equations involving powers. For example, take both sides of the equation $x^3 = 125$ to the power $\frac{1}{3}$ to find the solution $x = 5$.

Example 2 Solving for the Base in a Power

Solve for x. Assume x is positive.

a) $x^4 = 16$ **b)** $x^{\frac{3}{2}} = 27$

Solution

Use inverse operations to "undo" the exponents.

a) $x^4 = 16$ Raise both sides to the exponent $\frac{1}{4}$.

$$\begin{aligned} (x^4)^{\frac{1}{4}} &= 16^{\frac{1}{4}} \\ x &= \sqrt[4]{16} \\ &= 2 \end{aligned}$$

By the power of a power rule, $(x^4)^{\frac{1}{4}} = x^{4 \times \frac{1}{4}} = x^1 = x$.

To check, substitute $x = 2$ in $x^4 = 16$.

L.S.	R.S.
$(2)^4 = 16$	16

L.S. = R.S., so the solution is correct.

b) $x^{\frac{3}{2}} = 27$ Raise both sides to the exponent $\frac{2}{3}$.

$$\begin{aligned} (x^{\frac{3}{2}})^{\frac{2}{3}} &= 27^{\frac{2}{3}} \\ x &= (\sqrt[3]{27})^2 \\ &= 3^2 \\ &= 9 \end{aligned}$$

By the power of a power rule, $(x^{\frac{3}{2}})^{\frac{2}{3}} = x^{\frac{3}{2} \times \frac{2}{3}} = x^1 = x$.

Example 3 Solving a Financial Problem

Materials

- scientific calculator

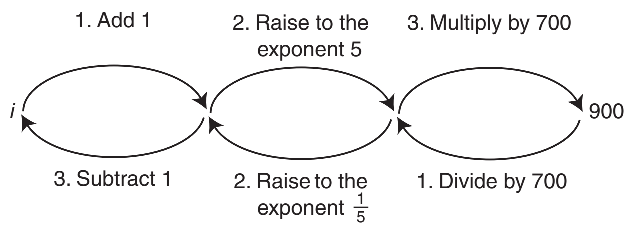
Under annual compounding, a principal of \$700 grows to \$900 in 5 years. Determine the annual interest rate.

Solution

Use the formula for compound interest: $A = P(1 + i)^n$. Substitute: $A = 900$, $P = 700$, and $n = 5$ to obtain

$900 = 700(1 + i)^5$

Draw an arrow diagram to determine how to isolate i.



$900 = 700(1 + i)^5$	Divide each side by 700.
$\frac{900}{700} = (1 + i)^5$	Raise each side to the exponent $\frac{1}{5}$.
$\left(\frac{900}{700}\right)^{\frac{1}{5}} = (1 + i)$	Evaluate the left side.
$1.0515 \doteq 1 + i$	Subtract 1 from each side.
$0.0515 \doteq i$	

The interest rate is approximately 5.15%.

Practice

A

- For help with questions 1, 2, and 6, see Example 1.

1. Determine each value without using a calculator.

a) $36^{\frac{1}{2}}$ b) $81^{\frac{1}{2}}$

c) $144^{\frac{1}{2}}$ d) $0.25^{\frac{1}{2}}$

2. Determine each value without using a calculator.

a) $8^{\frac{1}{3}}$ b) $64^{\frac{1}{3}}$

c) $0.027^{\frac{1}{3}}$ d) $(-125)^{\frac{1}{3}}$

3. Rewrite each expression using rational exponents.

a) $\sqrt{64}$ b) $\sqrt{1.21}$

c) $\sqrt[3]{216}$ d) $\sqrt[3]{-343}$

4. Determine the value of each expression in question 3.

5. Determine the value of each expression.

a) $\sqrt[4]{16}$ b) $\sqrt[4]{0.0256}$

c) $\sqrt[5]{-243}$ d) $\sqrt[6]{64}$

B

6. Write each expression in radical form, and then evaluate without a calculator.

a) $243^{\frac{1}{5}}$ b) $9^{\frac{3}{2}}$ c) $8^{\frac{5}{3}}$ d) $81^{\frac{3}{4}}$

e) $0.0625^{\frac{1}{4}}$ f) $(-32)^{\frac{3}{5}}$ g) $0.01^{\frac{3}{2}}$ h) $(-27)^{\frac{4}{3}}$

7. The expression $a^{\frac{m}{n}}$ can be interpreted as $(\sqrt[n]{a})^m$ or $\sqrt[n]{a^m}$.

a) Evaluate $16^{\frac{3}{2}}$ as $(\sqrt{16})^3$.

b) Evaluate $16^{\frac{3}{2}}$ as $\sqrt{16^3}$.

c) Which form did you find easier to evaluate? Explain.

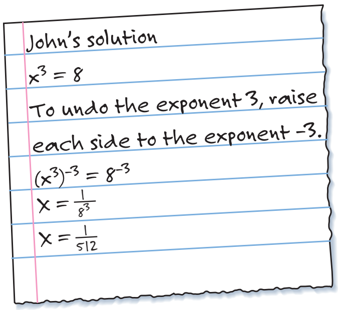
8. Scientists use the formula $D = 0.099M^{\frac{9}{10}}$ to give the drinking rate, D litres per day, for a mammal with mass M kilograms.

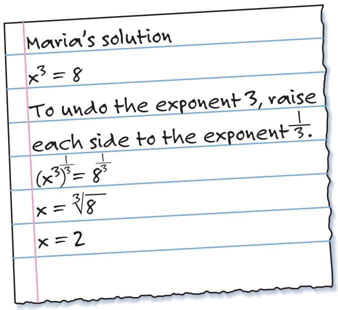
a) Rewrite the equation using radicals.

b) Determine the drinking rate of each mammal.

i) a 35-kg dog ii) a 520-kg moose iii) a 28-g mouse

9. John and Maria are comparing their solutions to the equation $x^3 = 8$. Whose solution is correct? Justify your answer.





■ For help with question 10, see Example 2.

10. Solve for x. Assume x is positive.
How do you know that your answers are correct?

a) $x^{\frac{1}{2}} = 7$ **b)** $x^2 = 9$ **c)** $x^3 = 64$

d) $x^{\frac{3}{2}} = 8$ **e)** $x^3 = \frac{27}{64}$ **f)** $x^{\frac{4}{3}} = 625$

■ For help with question 11, see Example 3.

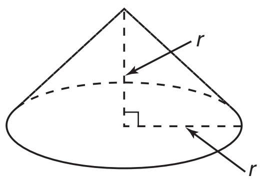
11. Determine the annual interest rate needed to double the value of a \$500 investment in 7 years. Assume that the interest is compounded annually.

12. Honeybees came to North America in the early 1600s with English settlers. In one region, the area, A hectares, inhabited by honeybees increased according to the formula $A = 0.5(9)^{\frac{t}{2}}$, where t is the number of years since introduction.

a) Determine the area inhabited by honeybees after 1 year.

b) Determine the area inhabited by honeybees after 3 years.

13. The equation $V = \frac{1}{3}\pi r^3$ gives the volume, V, of a cone whose height and base radius, r, are equal. Determine the radius of the cone if its volume is 1000 cm^3.



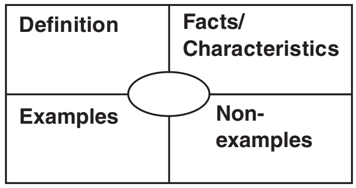
14. **Assessment Focus** The brain mass and body mass of mammals are approximately related by the formula $b = 0.011m^{\frac{2}{3}}$. In the formula, b is the brain mass in kilograms and m is the body mass in kilograms.

a) Determine the brain mass of a 512-kg giraffe.

b) Determine the brain mass of a 420-g chinchilla.

c) The brain mass of a cat is about 0.025 kg. Determine its body mass. Explain your strategy.

15. **Literacy in Math** Use a Frayer Model or a graphic organizer of your choice. Explain what a rational exponent is and how to simplify an expression involving a rational exponent. Use examples in your explanation.



C **16.** The formula $P = \frac{r^2s^3}{2}$ gives the approximate power, P watts, generated by a wind turbine with radius r metres when the wind speed is s metres per second.

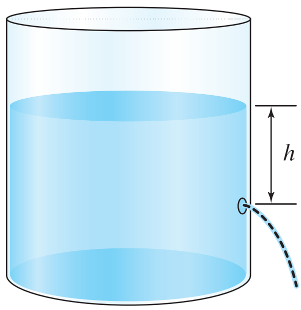
a) Rearrange the formula to isolate r. Give your answer in rational exponent form and radical form.

b) Repeat part a for s.

17. The speed, s metres per second, at which a liquid flows from a small hole in a container is given by the formula $s \doteq (19.6h)^{\frac{1}{2}}$, where h metres is the height of the liquid above the hole.

a) Determine the speed at which the liquid flows when the liquid is 1.0 m above the hole.

b) What height corresponds to a flow speed of 2 m/s? Round your answer to the nearest centimetre.



In Your Own Words

Explain what a rational exponent represents.
Describe how rational exponents can help you solve equations.
Explain how to decide which rational exponent to use in solving the equation. Include examples in your explanation.

GAME

Power Dominoes

Materials

- 15 power domino tiles

Play with a partner.

- Shuffle the power domino tiles.
  Spread them out face down.
- Each player takes seven tiles.
  Turn the remaining tile face up.
- Players take turns matching an end of one of their tiles to an end of a tile on the table. Tile ends match if they simplify to the same expression. For example, a^2a^{-4} matches $\frac{a^3}{a^5}$ since both expressions simplify to a^{-2}.

$r^3\ r^{-1}$	$a^2\ a^{-4}$	$\frac{a^3}{a^5}$	$\frac{1}{r^{-3}}$

- If a player cannot make a match or makes an incorrect match, play passes to the other player.
- The player who uses all seven tiles first wins.

Reflect

- What is a pair of expressions that does not simplify to the same expression? Tell how you know.

6.6 Exponential Equations

Salmonella is a bacterium that causes food poisoning. Under favourable conditions, it takes 1 salmonella bacterium about 20 min to divide into 2 new salmonella.

Investigate — Solving an Exponential Equation

Materials
- grid paper or graphing calculator
- scientific calculator

A lab technician starts with 1 salmonella bacterium. She uses the equation $P = 2^{3t}$ to model the population, P, of salmonella after t hours. To determine when there will be 1000 salmonella, she solves the **exponential equation** $1000 = 2^{3t}$.

In an exponential equation, the unknown is an exponent.

Work with a partner.

Three students are discussing how to solve the equation $2^{3t} = 1000$.

- Jawad suggests substituting different values for t in 2^{3t} until the correct value is obtained.
- Lily suggests using a graph of $P = 2^{3t}$.
- Max suggests isolating t by raising each side of the equation to the exponent $\frac{1}{3}$.

Will each of these strategies work?
Solve for t using each strategy that works.
Explain why the other strategy or strategies will not work.

Reflect

Compare the exponential equation $2^{3t} = 1000$ to the equations in *Example 2* of Lesson 6.5.

- How are the equations the same? How are they different?
- Is it possible to use the same strategy to solve both types of equations? Justify your answer.

Connect the Ideas

An exponential equation is an equation that contains a variable in the exponent. Some examples of exponential equations are:

$2^x = 32$ $\quad$ $9^{x+1} = 27^x$ $\quad$ $(0.8)^x = 0.18$

Without technology

Some exponential equations can be solved by writing both sides of the equation as powers of the same base. This allows us to use the following property.

Equality of powers with a common base
If $a^m = a^n$, then $m = n$ $(a > 0, a \neq 1)$

For example, since 4^x and 4^3 are both powers of 4, the solution to $4^x = 4^3$ is $x = 3$.

Example 1 Finding a Common Base

Solve.

a) $5^x = 5^6$

b) $2^x = 32$

c) $7^{3x-4} = 49$

d) $3^{5x+8} = 27^x$

e) $2^{2(x-5)} = 4^{3x-1}$

Solution

a) $5^x = 5^6$ $\quad$ Equate the exponents.
$x = 6$

b) $2^x = 32$ $\quad$ Write 32 as a power of 2.
$2^x = 2^5$ $\quad$ Equate the exponents.
$x = 5$

To check, substitute $x = 2$ in 7^{3x-4}.

L.S.	R.S.
7^{3x-4}	49
$= 7^{3(2)-4}$	
$= 7^2$	
$= 49$	

L.S. = R.S., so the solution is correct.

When we take a power of a power, we multiply the exponents. So $(3^3)^x = 3^{3x}$.

c) $7^{3x-4} = 49$ Write 49 as a power of 7.

$7^{3x-4} = 7^2$ Equate the exponents.

$3x - 4 = 2$ Solve for x.

$3x = 6$

$x = 2$

d) $3^{5x+8} = 27^x$ Write 27 as a power of 3.

$3^{5x+8} = (3^3)^x$ Simplify the right side.

$3^{5x+8} = 3^{3x}$ Equate the exponents.

$5x + 8 = 3x$ Solve for x.

$8 = -2x$

$-4 = x$

e) $2^{2(x-5)} = 4^{3x-1}$ Write 4 as a power of 2.

$2^{2(x-5)} = (2^2)^{3x-1}$ Simplify each side.

$2^{2x-10} = 2^{6x-2}$ Equate the exponents.

$2x - 10 = 6x - 2$ Solve for x.

$-4x = 8$

$x = -2$

With technology

Most exponential equations cannot be easily expressed as powers of the same base. We use technology to solve these equations.

Example 2 Using Systematic Trial

Materials

- scientific calculator

Use systematic trial to solve $3^x = 7$ to 2 decimal places.

Solution

7 is between $3^1 = 3$ and $3^2 = 9$, but closer to 9.

So, the solution to $3^x = 7$ is between 1 and 2, but closer to 2.

Try $x = 1.6$: $3^{1.6} \doteq 5.80$ (too small)

Try $x = 1.7$: $3^{1.7} \doteq 6.47$ (still too small)

Try $x = 1.8$: $3^{1.8} \doteq 7.22$ (too large)

Try $x = 1.78$: $3^{1.78} \doteq 7.07$ (still too large)

Try $x = 1.77$: $3^{1.77} \doteq 6.99$ (close enough)

So, $x \doteq 1.77$.

Example 3

Using a Graph

Materials

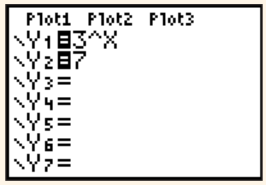
- TI-83 or TI-84 graphing calculator

Use a graph to solve $3^x = 7$ to 2 decimal places.

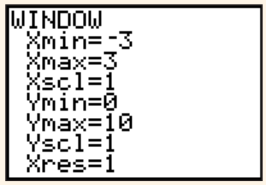
Solution

Graph $y = 3^x$ and $y = 7$ on the same screen, and determine the x-coordinate of the point of intersection.

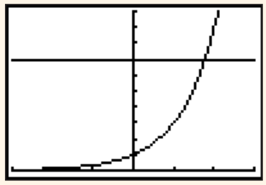
Enter the equations.
Press [Y=].
At Y_1 =, press 3 [^] [X,T,Θ,*n*].
At Y_2 =, press 7.



Set the viewing window.
Press [WINDOW].
Change the settings to those shown at the right.

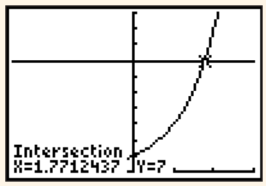


Graph the equations.
Press [GRAPH].
The graph of $y = 3^x$ is the curve. The graph of $y = 7$ is the horizontal line.



Determine the x-coordinate of the point of intersection.

- Use the INTERSECT feature in the CALC menu.
  Press [2nd] [TRACE] 5.
- At each prompt, press [ENTER].
- The x-coordinate of the point of intersection is $x \doteq 1.7712437$.



To the nearest hundredth, the solution is $x \doteq 1.77$.

Exact versus approximate solutions

Compare the answers in *Example 1* to the answer in *Examples 2* and *3*. The solutions to $2^x = 32$, $7^{3x-4} = 49$, and $3^{5x+8} = 27^x$ are exact since 32, 49, and 27 are powers of 2, 7, and 3 respectively. We can only approximate the solution of $3^x = 7$ since 7 is not a power of 3.

Practice

A

■ For help with questions 1 and 5, see Example 1.

1. Solve each equation.

a) $4^x = 4^3$ b) $7^x = 7^2$ c) $2^x = 2^7$ d) $5^{2x} = 5^3$

2. Solve each equation.

a) $x - 8 = 7$ b) $4x + 1 = 9$

c) $11 - 2x = 5 + x$ d) $2(x - 6) = 3x$

3. Solve each equation.

a) $3^{x+3} = 3^8$ b) $10^{x-3} = 10^{-2}$

c) $2^{3x} = 2^{8-x}$ d) $6^{3x-7} = 6^{x+2}$

4. Express each number as a power.

a) 36 as a power of 6 b) 16 as a power of 2

c) 125 as a power of 5 d) 1000 as a power of 10

5. Express the right side of the equation as a power of 3, then solve the equation.

a) $3^x = 9$ b) $3^x = \frac{1}{9}$ c) $3^{2x} = 81$ d) $3^{x+5} = 27$

How do you know that your answers are correct?

B

6. a) Solve $2^x = 16$. Explain your strategy.

b) Can you use the same strategy to solve $2^x = 25$? Explain.

7. Solve each equation algebraically.

a) $5^x = 125$ b) $4^{2x} = 64$

c) $2^{x+1} = 8$ d) $6^{x-1} = 36$

e) $7^{2x-1} = 49$ f) $10^{1-2x} = 100$

g) $3^{x+1} = \frac{1}{9}$ h) $2^{3x+6} = 1$

8. Choose two equations from question 7. Explain the steps in the solution. Check your solution by substituting for x.

■ For help with question 9, see Example 2.

9. a) Use the base of the power on the left side of each equation. Between which two integer powers of the base does the solution lie? Justify your answers.

i) $2^x = 30$ ii) $5^x = 100$ iii) $3^x = 75$ iv) $2^x = \frac{1}{5}$

b) Use systematic trial to solve each equation in part a. Round to 2 decimal places.

10. Solve each equation algebraically.

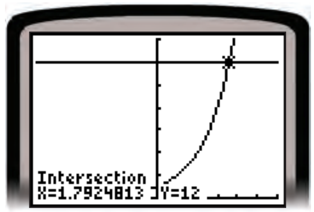
a) $9^{x+1} = 27^x$ b) $4^{3x} = 32^{x-1}$ c) $3^{2(x+2)} = 27^{x+2}$

d) $7^{3x-5} = 49^{-x}$ e) $100^{2x-3} = 1000^{3x+1}$ f) $5^{2(x-5)} = 125^{x-1}$

11. Choose two equations from question 10. Explain your choice of strategy.

■ For help with question 12, see Example 3.

12. Use the graph of $y = 4^x$ to solve the equation $4^x = 12$. Describe the steps you used to find the solution. Verify your answer numerically.



13. Determine the approximate solutions using graphing technology.
Round to 2 decimal places.

a) $3^x = 14$ b) $7^x = 100$ c) $10^x = 50$

d) $2^x - 36 = 0$ e) $3^{2x} = 300$ f) $(1.06)^x = 2$

Include a sketch of the graphing calculator screen in your solution.

14. Consider the equation $5^{2x-1} = 45$.

a) Will the solution be exact or approximate? Justify your answer.

b) Solve the equation. Explain your choice of strategy.

15. A strain of bacteria doubles every hour. A lab technician starts with 100 bacteria. He uses the equation $B = 100(2)^t$ to model the number of bacteria, B, after t hours.

a) Write an exponential equation that can be used to determine when there are 6400 bacteria in the culture.

b) Solve the equation. Explain your choice of strategy.

16. **Assessment Focus**

a) Solve each equation algebraically: i) $2^{2x+3} = 8$ ii) $9^x = 27^{-x+2}$

b) Explain how you solved the equation $2^{2x+3} = 8$.
How do you know that your answer is correct?

c) Solve $9^x = 27^{-x+2}$ graphically.
Include the graphing calculator screen in your solution.
How is the solution related to the graphs of $y = 9^x$ and $y = 27^{-x+2}$?
Explain.

17. **Literacy in Math** List all of the strategies you have used in this chapter to solve equations. Include an example for each strategy. Use a graphic organizer to present your work.

C **18.** The planning department of a township is responsible for estimating the number of utility hook-ups needed in new subdivisions. The current capacity is about 729 hook-ups per year. The actual demand, D, for hook-ups is given by $D = 27(3^n)$, where n is the number of years since 2007.

a) Use algebra to determine when the demand reaches the capacity.

b) Use graphing technology to verify your answer.

19. A cross-country skier forgets a mug of coffee and a muffin in a snowbank. Their temperatures, in degrees Celsius, after t minutes can be modelled by the formulas:

$T_{\text{coffee}} = 81 \times 3^{-2t}$ and $T_{\text{muffin}} = 27 \times 3^{-t}$

a) Use algebra to determine when the coffee cools to the same temperature as the muffin. What is the common temperature?

b) Verify your answer by using graphing technology.

In Your Own Words

You have learned several strategies for solving exponential equations. How do you decide which strategy to use in a given situation? Include examples in your explanation.

6.7 Applications of Exponential Equations

Carbon-14 (C-14) is a radioactive element that is absorbed from the atmosphere by plants and animals while they are alive. When a plant or animal dies, the C-14 in the organism's remains decays exponentially over time.

Investigate

Modelling Carbon-14 Decay

Materials
- container with a lid
- paper plate
- 100 pennies
- 100 nickels
- grid paper or graphing calculator

C-14 atoms decay to nitrogen atoms. We use pennies to represent C-14 atoms and nickels to represent nitrogen atoms.

Each penny that lands heads up represents a C-14 atom that has decayed into a nitrogen atom.

The symbol for microgram is µg. 1000 µg = 1 mg

Approximately every 5730 years, the amount of C-14 in the remains of an organism is reduced by a factor of one-half. Scientists can estimate when the organism was alive by comparing the amount of C-14 in the remains to the amount of C-14 in a living organism.

Work in a group of 4.

- Place 100 pennies in the container. Copy this table.

Trial number	Number of C-14 atoms
0	100

- Shake the container and empty it onto the paper plate. Replace each penny that lands heads up with a nickel. Then, record the trial number and the number of pennies left on the plate.
- Pour the coins on the plate into the container. Repeat the previous step for 4 more trials.
- Work individually. Graph the data in the table.
- Explain why each trial represents a time span of approximately 5730 years.
- Suppose a bone that originally contained 100 µg of C-14 now contains 10 µg of C-14. Estimate the age of the bone. Justify your answer.

Reflect

- What fraction of the pennies would you expect to remain after each trial? How closely did your results match this expected result? Explain.
- Did you use the table or graph to estimate the age of the bone? Explain the reason for your choice.
- C-14 dating is only used for objects less than 50 000 years old. Use your table or graph to explain why.

Connect the Ideas

Solving $y = ab^x$ for x

Exponential relations can be modelled by $y = ab^x$.

a is the initial value

b is the growth/decay factor

- $b > 1$ models growth
- $0 < b < 1$ models decay

Real-world applications of exponential growth or decay may require solving the equation $y = ab^x$ for x.

Example 1 Using a Table of Values

Materials

- TI-83 or TI-84 graphing calculator

The variable t is often used to represent time.

The population of Ontario was 9.3 million in 1985 and has been growing at an annual rate of 1.5%. This situation can be modelled by the equation $P = 9.4(1.0125)^t$, where P million represents the population t years after 1985. In which year did Ontario's population first exceed 10 million?

Solution

Use the equation $P = 9.4(1.0125)^t$.

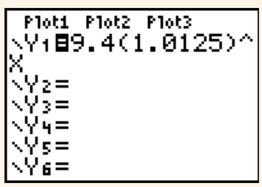
Substitute: $P = 10$

$10 = 9.4(1.0125)^t$

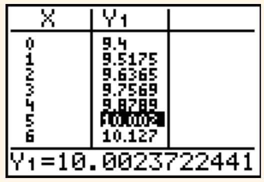
Use a table of values to solve for t.

Use the variable X on the calculator to represent t in the equation.

Enter the right side of the equation in the equation editor. Press [Y=].
Move the cursor to Y_1=.
Press 9.4 [(] 1.0125 [)] [^] [X,T,Θ,n].



$t = 0$ corresponds to the year 1985.

Set up the TABLE editor. Press [2nd] [WINDOW]. Set TblStart $= 0$ so that the table starts at $X = 0$. Set ΔTbl $= 1$ so that X increases in steps of 1.	
Generate the table. Press [2nd] [GRAPH]. Scroll through the table until Y_1 is greater than 10. This occurs at $X = 5$.	

The population first exceeds 10 million approximately five years after 1985; that is, in the year 1990.

Example 2

Using a Graph

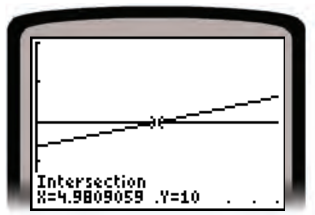
Materials

- TI-83 or TI-84 graphing calculator

Redo *Example 1*.
Use a graph to solve for t.

Solution

We wish to solve $10 = 9.4(1.0125)^t$ for t.
Graph the left and right sides of the equation on the same screen, and determine the X-coordinate of the point of intersection.



At the point of intersection, $X \doteq 5$. The population exceeds 10 million five years after 1985; that is, in the year 1990.

Exponential equations involving doubling time and half-life

In Chapter 5, you learned that quantities that grow or decay exponentially increase or decrease at a constant percent rate. These quantities have a constant doubling time or half-life. When the doubling time, d, or half-life, h, is known, the relationship between the initial amount, A_0, and the amount A after time t can be modelled by these equations.

Exponential growth

$$A = A_0(2)^{\frac{t}{d}}$$

Exponential decay

$$A = A_0(0.5)^{\frac{t}{h}}$$

Example 3

Solving an Application Involving Half-Life

Materials

- TI-83 or TI-84 graphing calculator

Caffeine has a half-life of approximately 5 h. Suppose you drink a cup of coffee that contains 200 mg of caffeine. How long will it take until there is less than 10 mg of caffeine left in your bloodstream? Give your answer to 1 decimal place.

Solution

Every 5 h, the amount of caffeine in your bloodstream is reduced by a factor of one-half.

Use the equation $A = A_0(0.5)^{\frac{t}{h}}$.

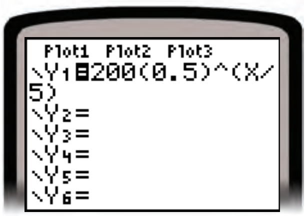
Substitute: $A = 10$, $A_0 = 200$, and $h = 5$

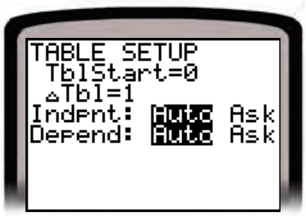
$10 = 200(0.5)^{\frac{t}{5}}$

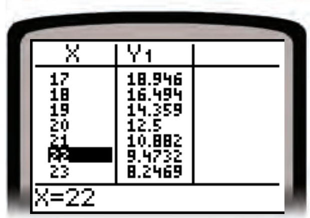
Create a table of values to solve for t.

Scroll down until Y_1 is less than 10.

You could also use a graph to solve for *t*.

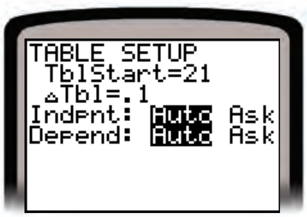


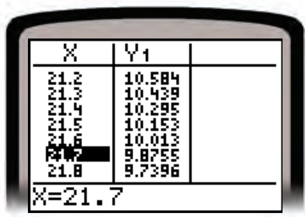




Y_1 is greater than 10 at X = 21 and less than 10 at X = 22.

To find the solution to the nearest tenth, go back to the table setup and change it so that X starts at 21 and increases by 0.1.





To the nearest tenth, X ≐ 21.7.

After 21.7 h, there is less than 10 mg of caffeine left in your bloodstream.

Practice

A

For help with questions 1 and 2, see Example 1.

1. A new car decreases in value exponentially after it is purchased. The value, V dollars, of a certain car t years after it was purchased is given by $V = 20\,000(0.84)^t$. Write an exponential equation that can be used to determine when the value of the car is equal to each amount.

a) \$10 000 **b)** \$15 000

c) \$7500 **d)** \$17 500

2. Use the table to determine in which year the value of the car will first be less than each amount in question 1.

X	Y1
0	20000
1	16800
2	14112
3	11854
4	9957.4
5	8364.2
6	7026

X=6

3. The population, P million, of Alberta between 1987 and 2005 can be modelled by the equation $P = 2.4(1.017)^t$, where t is the number of years since 1987. Write an exponential equation that can be used to estimate when the population equalled each number of people.

a) 2.5 million **b)** 2.7 million **c)** 3.0 million

4. Use the tables to estimate in which year the population first exceeded each number of people in question 3.

X	Y1
1	2.4408
2	2.4823
3	2.5245
4	2.5674
5	2.6111
6	2.6554
7	2.7006

X=7

X	Y1
8	2.7465
9	2.7932
10	2.8407
11	2.889
12	2.9381
13	2.988
14	3.0388

X=14

5. Match each equation with its table of values.

a) $y = 4^x$ **b)** $y = 3(4)^x$ **c)** $y = 3(2)^x$

i)

X	Y1
0	3
1	6
2	12
3	24
4	48
5	96
6	192

X=0

ii)

X	Y1
0	1
1	4
2	16
3	64
4	256
5	1024
6	4096

X=0

iii)

X	Y1
0	3
1	12
2	48
3	192
4	768
5	3072
6	12288

X=0

6. Use the table of values.

a) What is the value of y for each value of x?

i) $x = 2$ **ii)** $x = 5$

b) What is the value of x for each value of y?

i) $y = 2$ **ii)** $y = 0.5$

X	Y_1
0	.25
1	.5
2	1
3	2
4	4
5	8
6	16

X=0

B

7. The table shows the growth of a culture of bacteria over time under laboratory conditions. The variable X represents the time in hours and the variable Y_1 represents the number of bacteria.

a) How many bacteria were present initially? How do you know?

b) How long does it take for the population to double? Justify your answer.

X	Y_1
0	100
1	141.42
2	200
3	282.84
4	400
5	565.69
6	800

X=0

8. A principal of \$500 is invested at 8% per year, compounded annually. After n years, the amount of the investment, A dollars, is given by $A = 500(1.08)^n$. Write an exponential equation that can be used to determine how long it takes for the investment to:

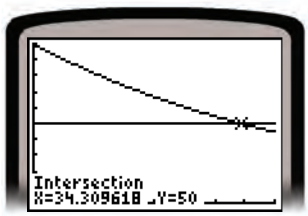
a) grow to \$600 **b)** double in value **c)** triple in value

■ For help with questions 9 and 10, see Example 2.

9. Blue jeans fade with repeated washing. The equation $P = 100(0.98)^n$ models the percent, P, of colour left after n washings.

a) Write an exponential equation that can be used to determine the number of washings until 50% of the colour remains.

b) Use the graph to solve the equation. Justify your answer.



10. Lupine is a wildflower that attracts honeybees and butterflies. The equation $N = 100(1.4)^t$ models the number, N, of wild lupine seeds in a wildflower seed bank t years after collection began.

a) Write an exponential equation that can be used to estimate when there will be 2000 seeds in the bank.

b) Use graphing technology to solve the equation in part a. Include a sketch of the graphing calculator screen in your solution.

For help with question 11, see Example 3.

11. Suppose you invest \$500 at 6% per year, compounded annually. The value, A dollars, of your investment after n years is given by $A = 500(1.06)^n$.

a) Use graphing technology to graph $A = 500(1.06)^n$.

b) Estimate the number of years that it will take for your investment to grow to each amount.

i) \$600 **ii)** \$1000 **iii)** \$1200

Use the formula $A = A_0(0.5)^{\frac{t}{h}}$ when you know the half-life.

12. The mortality rate from heart attack can be modelled by the relation $M = 88.9(0.9418)^t$, where M is the number of deaths per 100 000 people and t is the number of years since 1998.

a) Has the mortality rate increased or decreased since 1998? Justify your answer.

b) When will the mortality rate be one-half the rate in 1998?

c) When will the mortality rate decrease to 22.2 deaths per 100 000? Justify your answers.

13. **Assessment Focus** Dye is injected to test pancreas function. The mass, R grams, of dye remaining in a healthy pancreas after t minutes is given by $R = I(0.96)^t$, where I grams is the mass of dye initially injected.

a) Suppose 0.50 g of dye is injected into a healthy pancreas. How long will it take until 0.35 g of dye remain? Justify your answer.

b) Describe the steps used to solve part a.

c) Find the half-life of the dye in a healthy pancreas.

d) How would the half-life change for a patient with an overactive pancreas? Explain.

14. Airplane cabins are pressurized because air pressure decreases as the height above sea level increases. The equation $P = 100(0.917)^h$ models the air pressure, P kilopascals, at a height of h kilometres above sea level.

a) Determine the air pressure at a height of 10 km.

b) At what altitude is the air pressure 50% of its value at sea level?

c) For which part did you have to solve an exponential equation? Explain.

15. Tritium, a radioactive gas that builds up in CANDU nuclear reactors, is collected, stored in pressurized gas cylinders, and sold to research laboratories. Tritium decays into helium over time. Its half-life is about 12.3 years.

a) Write an equation that gives the mass of tritium remaining in a cylinder that originally contained 500 g of tritium.

b) Estimate the time it takes until less than 5 g of tritium is present.

16. An archaeologist uses radiocarbon dating to determine the age of a Viking ship. Suppose that a sample that originally contained 100 mg of Carbon-14 now contains 85 mg of Carbon-14. What is the age of the ship to the nearest hundred years?

Use the formula $A = A_0(2)^{\frac{t}{d}}$ when you know the doubling time.

17. A colony of bacteria doubles in size every 20 min. How long will it take for a colony of 20 bacteria to grow to a population of 10 000?

18. Compare the formula for doubling time to the formula for half-life. How are they the same? How are they different? Explain.

19. The formula $P = 29.6(1.0124)^t$ models Canada's population, where P is the population in millions and t is the number of years since 1995.

a) Determine the doubling time for Canada's population.

b) Use the result of part a and the formula $A = A_0(2)^{\frac{t}{d}}$ to model the growth of Canada's population in terms of its doubling time.

c) Use both models to determine when Canada's population will first reach 40 million. Why should the answers be the same?

20. **Literacy in Math** The words *exponent, exponential growth, exponential decay, exponential relation, exponential regression,* and *exponential equation* are used when working with real-world situations involving growth and decay. Define each term and give an example of it.

C

21. The exponential relations $P = 100(0.87)^t$ and $P = 100(0.5)^{\frac{t}{5}}$ can be used to model the percent of caffeine in your bloodstream t hours after you drink a beverage containing caffeine.

a) What do the numbers in the relation $P = 100(0.87)^t$ represent?

b) What do the numbers in the relation $P = 100(0.5)^{\frac{t}{5}}$ represent?

c) Explain why the expression $(0.5)^{\frac{t}{5}}$ can be rewritten as $[(0.5)^{\frac{1}{5}}]^t$. Evaluate $(0.5)^{\frac{1}{5}}$.

d) Explain how the result of part c shows that the relations $P = 100(0.5)^{\frac{t}{5}}$ and $P = 100(0.87)^t$ are equivalent.

22. When a nuclear reactor is shut down, the core contains many radioactive isotopes which continue to decay. These exponential relations model the activity, A becquerels, for two isotopes after t days.

Iodine-131: $A_I = (3 \times 10^{18}) \times (0.5)^{\frac{t}{8}}$, Xenon-133: $A_{Xe} = (6 \times 10^{18}) \times (0.5)^{\frac{t}{5}}$

a) Which isotope has the greater initial activity? Which decays more quickly? Explain.

b) After how many days will the two isotopes have the same activity?

c) Predict what will happen to the activities of the two isotopes after the time you found in part b.

In Your Own Words

Explain the difference between an exponential relation and an exponential equation. Explain how an exponential equation can be used to solve a situation that can be modelled by an exponential relation. Include an example in your explanation.

Home Quit

6.8 Occupations Using Mathematical Modelling

Many careers require you to be able to create and use graphical and algebraic models of real-world situations.

Inquire

Researching Careers Involving Mathematical Modelling

Materials
- computer with Internet access

Mathematical modelling is used in a variety of careers. For example:

- The manager of a timber company may use a mathematical model to determine the best age at which to harvest a stand of trees.
- A landscaper may use mathematical formulas to estimate the materials and labour needed to complete a project.
- A workplace safety officer may use a mathematical model to determine the effects of exposure to airborne asbestos.
- A sales manager may use graphical models to analyse trends and patterns in sales data.

Work in a small group.

Part A: Creating a List of Careers

Create a list of careers that use tables, graphs, or formulas to model relationships.

- Brainstorm to start the list.
  Briefly describe how mathematical modelling could be used in each career.
- Continue the list by going through each lesson in Chapters 5 and 6.
  List the careers referred to in the lessons.
  Briefly describe how mathematical modelling could be used in each career.

- Each member of the group should select two careers from the list to research further.

Part B: Researching Careers

- Investigate the careers you chose in Part A.
  You may choose to:
  - Read about the career on the Internet or in printed material.
  - Contact and interview someone who works in the career.
- If you use the Internet, you may type phrases like these into a search engine such as Google or Yahoo to help you start your research.

Search words
- nursing formulas
- media planner math
- police math modelling

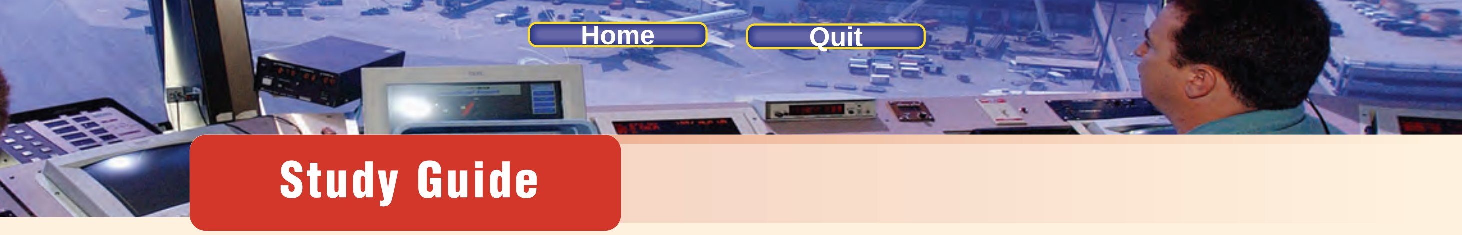
- Summarize your research.
  Include details that answer questions such as:
  - How is mathematical modelling used in this career?
    What is each model supposed to simulate, describe, or predict?
  - What technology is used to graph and analyse data or to evaluate formulas?
  - Is data collected to develop and test models?
  - What measurement system (metric and/or imperial) is used in this career? How important are accurate measurements and calculations?
  - What is a typical wage or salary for someone in this career?
    Are employees paid on an hourly or a salary basis?

Part C: Researching Educational Requirements

- Research the educational requirements for the careers you investigated in Part B.
  You may choose to:
  • Go to the Web sites of community colleges or other post-secondary institutions.
  • Use course calendars of post-secondary institutions or other information available through your school's guidance department.
- Record your findings.
- Be prepared to present your research from Parts A and B to the class.

Reflect

- What other areas of mathematics would likely be applied in a career that uses modelling?
- What types of written or graphical communication are used in these careers?
- What opportunities for advancement, leadership, or self-employment exist in these careers?



Study Guide

Rearranging Formulas

Rearrange the formula $V = \frac{4}{3}\pi r^3$ to isolate r.

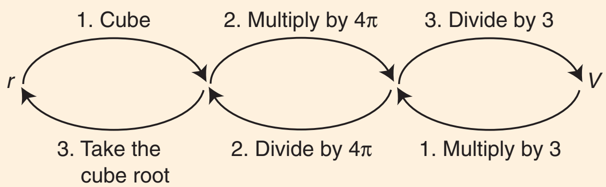
$V = \frac{4}{3}\pi r^3$ Multiply each side by 3.

$3V = 4\pi r^3$ Divide each side by 4π.

$\frac{3V}{4\pi} = r^3$ Take the cube root of each side.

$\sqrt[3]{\frac{3V}{4\pi}} = r$

Use inverse operations and a balance strategy.



Exponents

Definitions		Examples
• Positive integer exponents	$a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ factors}}$	$4^3 = 4 \times 4 \times 4 = 64$
• Zero exponents	$a^0 = 1,\ a \neq 0$	$4^0 = 1$
• Negative integer exponents	$a^{-n} = \frac{1}{a^n},\ a \neq 0$	$4^{-3} = \frac{1}{4^3} = \frac{1}{64}$
• Rational exponents	$a^{\frac{1}{n}} = \sqrt[n]{a},\ a > 0$ if n is even $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m},\ a > 0$ if n is even	$64^{\frac{1}{3}} = \sqrt[3]{64} = 4$ $64^{\frac{2}{3}} = (\sqrt[3]{64})^2 = 4^2 = 16$

Laws of Exponents		Examples
• Multiplication law	$a^m \times a^n = a^{m+n}$	$x^5 \times x^{-7} = x^{-2}$
• Division law	$a^m \div a^n = a^{m-n},\ a \neq 0$	$\frac{x^4y^4}{(x^2)^{-3}y^3} = \frac{x^4y^4}{x^{-6}y^3} = x^{4-(-6)}\,y^{4-3} = x^{10}\,y$
• Power of a power law	$(a^m)^n = a^{mn}$	$(y^4)^2 = y^8$

Solving Exponential Equations

Common base

Set the exponents equal to each other and solve the resulting equation.

$2^{x+3} = 32$

$2^{x+3} = 2^5$

$x + 3 = 5$

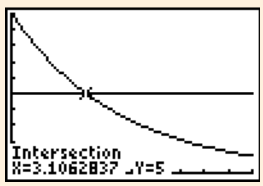
$x = 2$

Different bases

Solve for x: $10(0.8)^x = 5$

Plot $Y_1 = 10(0.8)^x$ and $Y_2 = 5$. Determine the X-coordinate of the point of intersection.

$x \doteq 3.1$



Chapter Review

6.1

1. Suzie works in a boutique. Each month, her earnings, E dollars, are given by the formula $E = 2500 + 0.10s$, where s dollars are her sales for the month. One month, Suzie's sales were \$8000. How much did she earn that month?

2. Bonnie wants to install a laminate floor in a room that measures 5.0 m by 3.0 m. The flooring comes in bundles. Each bundle costs \$47.76 and covers an area of 2.23 m^2. Determine the cost of the flooring.

3. Houd plans to replace the light bulbs in his house with energy-saver light bulbs. The formula $C = 0.006h + 0.45$ gives the cost, C dollars, of using a regular bulb for h hours. The cost for an energy-saver bulb is $C = 0.004h + 1.05$. How much money does Houd save with the energy saver bulb after 1000 h of use?

4. The formula $E = Pt$ gives the energy, E watt hours, used when an electrical appliance of power P watts is used for t hours. Determine the energy used in each situation.

a) A 17-W fluorescent bulb is left on for 1 h.

b) A 1400-W blow dryer is used for 10 min.

c) How long would the fluorescent bulb need to be left on so that it consumes as much energy as using the blow dryer for 10 min?

6.2

5. The circumference, C, of a circle with diameter d is $C = \pi d$.

a) Solve the formula for d.

b) Calculate the diameter of a circle with circumference 8 m.

6. The power in an electrical circuit is given by the formula $P = I^2R$, where P is the power in watts, I is the current in amperes, and R is the resistance in ohms. Determine R when $P = 1000$ W and $I = 12$ A. Explain your method.

7. The aspect ratio of a hang glider describes its performance during flight. The formula $R = \frac{s^2}{A}$, gives the aspect ratio, R, for a hang glider with wingspan s and wing area A.

a) Rearrange the formula to isolate s. Use an arrow diagram to show the inverse operations you can use.

b) Jake wants to design a hang glider with an aspect ratio of 2.7 and a wing area of 30 square feet. What will be the wingspan of the glider?

8. The formula $A = P(1 + rt)$ gives the amount, A dollars, of a simple interest investment. In the formula, P dollars is the principal invested, r is the annual interest rate expressed as a decimal, and t is the number of years.

a) Solve the formula for P.

b) The amount of a simple interest investment is \$2000 after 3.5 years. The principal earned 5% interest per year. Calculate the principal invested.

6.3 **9.** Evaluate without a calculator.

a) -5^2 b) 5^{-2} c) $(-5)^2$

Explain how you know whether the sign of the answer is positive or negative.

10. Simplify, then evaluate.

a) $10^2 \times 10^3$ b) $\dfrac{7^6}{7^{-2}}$

c) $(1.12^2)^3$ d) $\dfrac{2^3 \times 2^2}{2^5}$

For which parts did you use a calculator? Explain.

11. Simplify using the exponent rules.

a) $k^4l^3k^{-3}l^{-2}$ b) $(a^5)^{-2}(a^3)^4$

c) $\dfrac{xy^{-2}}{x^3y^2}$ d) $\left(\dfrac{s}{n}\right)^4\left(\dfrac{n}{s}\right)^{-3}$

12. Evaluate when $c = 5$ and $d = -3$.

a) c^d b) d^c

c) $\dfrac{cd^3}{c^3d}$ d) $(2cd^2)^3$

6.4 6.5 **13.** Evaluate.

a) $16^{\frac{1}{2}}$ b) $(-27)^{\frac{1}{3}}$ c) $32^{\frac{1}{5}}$

d) $8^{\frac{5}{3}}$ e) $625^{\frac{3}{4}}$ f) $0.81^{\frac{3}{2}}$

14. The formula $m = 3.5(0.5)^{\frac{t}{8}}$ gives the mass, m milligrams, of radioactive Iodine in a sample t days after the initial measurement. Determine the mass of radioactive Iodine after each number of days.

a) 2 days b) 4 days

c) 16 days

15. Solve for x. Assume x is positive. Show your work.

a) $x^2 = 100$ b) $x^3 = 64$

c) $x^{\frac{1}{2}} = 9$ d) $x^{\frac{3}{2}} = 8$

16. The formula $A = P(1 + i)^n$ gives the amount, A dollars, of a compound interest investment. In the formula, P dollars is the principal invested, i is the annual interest rate expressed as a decimal, and n is the number of years. Shelley invests \$500 with interest compounded annually. The amount of the investment after 6 years is \$651. Determine the annual interest rate.

17. The formula $B = 0.4089M^{\frac{3}{4}}$ gives the bird inhalation rate, B cubic metres of air per day, for a bird with mass M kilograms.

a) Rewrite the formula using radicals.

b) Calculate the inhalation rate for each bird.

i) a 4.5-kg bald eagle

ii) a 8.0-kg Canada goose

c) Determine the mass of a bird whose inhalation rate is twice that of the bald eagle.

d) Is the mass in part c twice that of the bald eagle? Explain.

6.6 **18.** Rewrite each set of numbers as powers of the same base.

a) 4, 16, 32

b) $\frac{1}{5}$, 25, 125

c) 9, 81, 243

19. Solve algebraically.

a) $4^{2x} = 4^6$

b) $5^x = 625$

c) $3^{2x+1} = 9$

d) $10^{x+1} = 10^{2x-3}$

e) $4^{3x-2} = 32^{x+1}$

f) $25^{x+1} = 125^{x-2}$

20. Choose 2 equations from question 19.

a) Explain how you solved the equations.

b) Use graphing technology to verify your answers to the equations in part a.

21. Determine the value of n to the nearest tenth. Explain your strategy.

a) $10^n = 125$

b) $3^n = 6$

c) $5^n = 0.01$

d) $250(1.03)^n = 400$

e) $1000(0.85)^n = 300$

22. Choose 2 equations from question 21. Verify your answers numerically.

6.7 **23.** Gillian takes a 300-mg tablet of pain medication. The amount, A milligrams, of medication remaining in her body after t hours is given by the formula $A = 300(0.8)^t$.

a) Write an exponential equation that can be used to determine when 1 mg of medication remains in Gillian's body.

b) Solve the equation in part a.

24. An archaeologist estimates the age of an artifact by measuring the amount of Carbon-14 it contains. The artifact originally contained 200 μg of Carbon-14. Now it contains 25 μg of Carbon-14. The half-life of Carbon-14 is about 5730 years. What is the age of the artifact to the nearest hundred years?

25. The resale value of a used vehicle is given by the formula $V = C(1 - r)^n$, where V dollars is the resale value, C dollars is the original price, r is the rate of depreciation as a decimal, and n is the age of the vehicle.

a) A certain vehicle depreciates at a rate of 20% per year. The original price of the vehicle was $36 000 and its resale value after n years is $7549.74. Write an exponential equation that can be used to determine the age of the vehicle.

b) Solve the equation in part a.

c) How do you know your answer in part b is correct?

6.8 **26.** Name two occupations where mathematical modelling is applied.

Home Quit

Practice Test

Multiple Choice: Choose the correct answer for questions 1 and 2. Justify each choice.

1. Which formula is equivalent to $E = \frac{mv^2}{2}$?

A. $m = \frac{2v^2}{E}$ **B.** $v = \sqrt{\frac{2E}{m}}$ **C.** $v = \sqrt{\frac{m}{2E}}$ **D.** $m = \frac{E}{2v^2}$

2. Which is the solution to $3^{2x+3} = \frac{1}{9}$?

A. $x = 0.5$ **B.** $x = 2$ **C.** $x = -0.5$ **D.** $x = -2.5$

Show your work for questions 3 to 6.

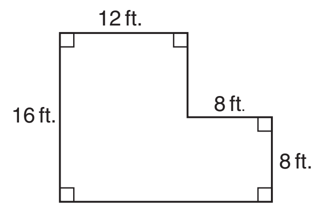
3. Knowledge and Understanding

a) Evaluate without a calculator.

i) $(-2)^{-3}$ **ii)** $\left(\frac{25}{9}\right)^{\frac{1}{2}}$ **iii)** $81^{\frac{3}{4}}$

b) Evaluate $\frac{2a^4b^{-3}}{a^2b}$ when $a = -5$ and $b = 2$.

4. Communication Sandy wants to estimate the cost of installing 2 layers of fibreglass insulation in an attic. The floor plan of the attic is shown. The insulation comes in batts that measure 23 inches by 47 inches. There are 10 batts in a bag, and a bag costs \$48.27. Make a plan for solving this problem, but *do not solve it*. Provide any formulas needed.



5. Application The population, P, of the city of Hazelton has grown according to the mathematical model $P = 32\ 000(1.09)^t$, where t is the number of years since 1990.

a) What do the numbers 32 000 and 1.09 represent?

b) If this trend continues, in what year will the population reach 100 000? Justify your answer.

6. Thinking A biologist uses the formula $E = km^{\frac{3}{4}}$ to model the relationship between the mass of a bird's egg, E grams, and the mass of the bird, m grams. The number k is a constant that is close to 0.25 for healthy eggs.

a) A ruby-throated hummingbird has a mass of 3.4 g. What is the mass of its egg?

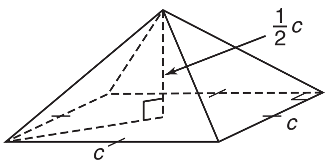
b) An ostrich egg has a mass of 1.4 kg. What is the mass of the mother bird?

c) A 7.3-kg whooping crane lays an egg whose mass is 208 g. Does this bird appear to be healthy? Explain.

Chapter Problem A Butterfly Conservatory

A glass butterfly conservatory is a square-based pyramid. The height of the conservatory is one-half the side length of the base. The volume is 900 m^3.

1. Demetra showed how the formula $V = \frac{1}{6}c^3$ gives the volume of the conservatory. Explain her thinking. Include your own diagram.



$$\begin{aligned} V &= \frac{1}{3}b^2h \\ &= \frac{1}{3}c^2 \times \frac{1}{2}c \\ &= \frac{1}{3} \times \frac{1}{2}c^2c \\ &= \frac{1}{6}c^3 \end{aligned}$$

2. **a)** Describe how to use a formula to determine the height and the side lengths of the base of the butterfly conservatory. Calculate these dimensions to the nearest tenth of a metre.

b) Substitute the dimensions into the formula to check. Are your results reasonable? Explain.

3. Describe a different way to determine the side lengths of the base and the height.

4. Suppose the height of the conservatory is doubled, but the base remains the same. Predict what would happen to the volume. Justify your prediction. Check your prediction. Explain how to extend your reasoning to make a generalization about the relationship between the height and the volume of a square-based pyramid.