Elastic Potential Energy and Simple Harmonic Motion 4.5

Imagine that you are to design a cord that will be used for bungee jumping from a bridge to a river (**Figure 1**). Although the distance between the bridge and the river is constant, the masses of the jumpers vary. The cord should offer complete and consistent safety, while at the same time providing a good bounce to prolong the thrill of jumping.

Figure 1
How would you test the properties of a bungee cord?

How could you analyze the force exerted by an elastic device, such as a bungee cord? What happens when the energy is transformed, as a person attached to the cord bounces up and down? A situation that involves the force changing as the cord is squeezed or compressed is more complex that the situations that we have looked at so far.

Hooke's Law

The force exerted by an elastic device varies as the device is stretched or compressed. To analyze the force mathematically, consider a horizontal spring attached to a wall and resting on a surface with negligible friction (**Figure 2**(**a**)). The position at which the spring rests, $x = 0$, is the *equilibrium position.* If a force is applied to the spring, stretching the spring to the right of equilibrium, the spring pulls back to the left as shown in **Figure 2**(**b**). Similarly, if a force is applied to compress the spring to the left of equilibrium, the spring pushes back to the right. In both cases, the direction of the force exerted by the spring is opposite to the direction of the force applied to the spring.

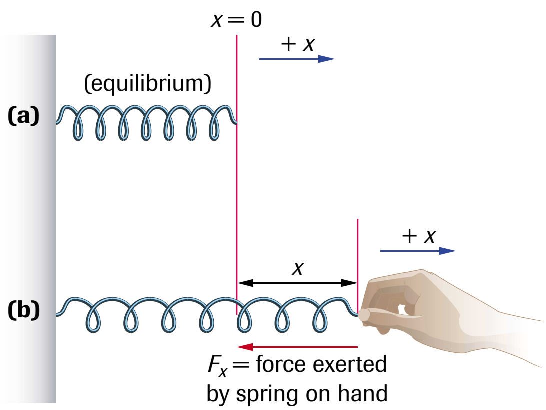


Figure 2
(a) A spring at its equilibrium position
(b) If the spring is stretched to the right, it exerts a force to the left: $F_x = -kx$.

Hooke's law the magnitude of the force exerted by a spring is directly proportional to the distance the spring has moved from equilibrium

ideal spring a spring that obeys Hooke's law because it experiences no internal or external friction

force constant (k) the proportionality constant of a spring

Experiments with springs show that the magnitude of the force exerted by the spring is directly proportional to the distance the spring has moved from equilibrium. This relationship is known as **Hooke's law**, after Robert Hooke (1635–1703), who published his law and its corresponding equation in 1678. Any spring that obeys Hooke's law is called an **ideal spring** because it experiences no friction, either internal or external. Using k as the constant of proportionality, we can write Hooke's law for the force exerted *by* a spring in equation form, in this case with horizontal components to correspond to the situation in **Figure 2(b)**:

$$F_x = -kx$$

where F_x is the force exerted by the spring, x is the position of the spring relative to equilibrium, and k (the proportionality constant) is the **force constant** of the spring. Springs that require a large force to stretch or compress them have large k values.

According to Hooke's law, if $x > 0$, then $F_x < 0$. In other words, if the spring is stretched in the $+x$ direction, it pulls in the opposite direction. Similarly, if $x < 0$, then $F_x > 0$, which means that if the spring is compressed in the $-x$ direction, it pushes in the opposite direction.

Since $-kx$ indicates the force exerted *by* the spring, we can apply Newton's third law to find that $+kx$ is the force applied *to* the spring to stretch or compress it to position x. Thus, Hooke's law for the force applied *to* a spring is:

$$F_x = kx$$

Although we have been referring to springs, Hooke's law applies to any elastic device for which the magnitude of the force exerted by the device is directly proportional to the distance the device moves from equilibrium.

LEARNING TIP

Forces Exerted *by* and Applied *to* a Spring

It is important to remember which force is exerted *by* the spring and which is applied *to* the spring. For example, a graph of the force exerted *by* the spring as a function of x has a negative slope. A graph of the force applied *to* the spring as a function of x has a positive slope. The magnitude of the force exerted *by* a spring is written either as $|kx|$ or as $k|x|$.

LEARNING TIP

Hooke's Law in General

To eliminate the need for new symbols, we use the same Hooke's-law equations for springs lying on an inclined plane or suspended vertically, as for springs in the horizontal plane. For example, in the vertical plane, $F = kx$ is the force applied to a spring, while the extension x is the change in the y position from the equilibrium position.

SAMPLE problem 1

A student stretches a spring horizontally a distance of 15 mm by applying a force of 0.18 N [E].

(a) Determine the force constant of the spring.

(b) What is the force exerted by the spring on the student?

Solution

(a) $F_x = 0.18$ N

$x = 15 \text{ mm} = 0.015 \text{ m}$

$k = ?$

Since the force is applied *to* the spring, we use the equation

$$F_x = kx$$

$$k = \frac{F_x}{x}$$

$$= \frac{0.18 \text{ N}}{0.015 \text{ m}}$$

$$k = 12 \text{ N/m}$$

The force constant is 12 N/m. (Notice the SI units of the force constant.)

(b) According to Newton's third law, if the force applied to the spring is 0.18 N [E], then the force exerted by the spring is 0.18 N [W].

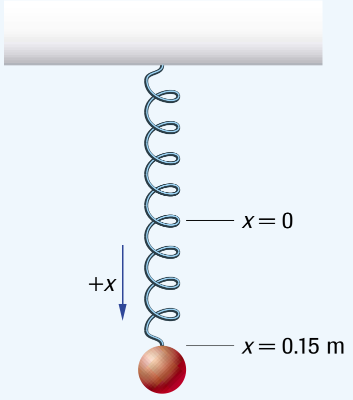
SAMPLE problem 2

A ball of mass 0.075 kg is hung from a vertical spring that is allowed to stretch slowly from its unstretched equilibrium position until it comes to a new equilibrium position 0.15 m below the initial one. **Figure 3(a)** is a system diagram of the situation, and **Figure 3(b)** is an FBD of the ball at its new equilibrium position.

(a) Determine the force constant of the spring.

(b) If the ball is returned to the spring's unstretched equilibrium position and then allowed to fall, what is the net force on the ball when it has dropped 0.071 m?

(c) Determine the acceleration of the ball at the position specified in (b).



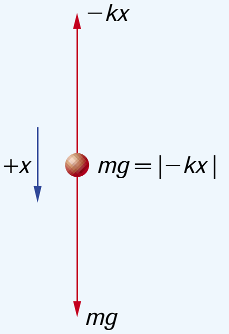


Figure 3
(a) The system diagram
(b) The FBD of the ball when the extension is 0.15 m

Solution

(a) We measure the extension x of the spring from its original unstretched position ($x = 0$) and choose $+x$ to be downward. Two vertical forces act on the ball: gravity and the upward force of the spring. At the new equilibrium position, the ball is stationary, so the net force acting on it is zero.

$m = 0.075$ kg

$x = 0.15$ m

$k = ?$

$$\sum F_x = 0$$
$$mg + (-kx) = 0$$
$$k = \frac{mg}{x}$$
$$= \frac{(0.075 \text{ kg})(9.8 \text{ N/kg})}{0.15 \text{ m}}$$
$$k = 4.9 \text{ N/m}$$

The force constant is 4.9 N/m.

(b) **Figure 4** is the FBD for the ball when $x = 0.071$ m. Considering the components of the forces in the vertical (x) direction:

$$\sum F_x = mg + (-kx)$$
$$= (0.075 \text{ kg})(9.8 \text{ N/kg}) - (4.9 \text{ N/m})(0.071 \text{ m})$$
$$\sum F_x = +0.39 \text{ N}$$

The net force is 0.39 N [down] when the ball has dropped to 0.071 m.

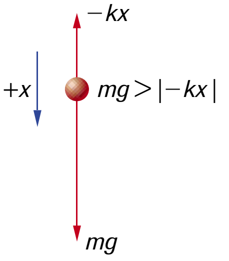


Figure 4
The FBD of the ball when the extension is 0.071 m

(c) $\sum F_y = 0.39$ N

$a_y = ?$

Applying Newton's second law:

$$\sum F_y = ma_y$$

$$a_y = \frac{\sum F_y}{m}$$

$$= \frac{0.39 \text{ N}}{0.075 \text{ kg}}$$

$$a_y = 5.2 \text{ m/s}^2$$

The acceleration is 5.2 m/s^2 [down] when the ball is at a spring extension of 0.071 m.

INVESTIGATION 4.5.1

Testing Real Springs (p. 220)
A graph of the force applied to an ideal extension spring suspended vertically yields a single straight line with a positive slope. How do you think a graph of the force applied to a real extension spring would compare?

In our applications of Hooke's law, we have assumed that the springs are ideal. To discover how real springs compare to ideal springs, you can perform Investigation 4.5.1 in the Lab Activities section at the end of this chapter.

Practice

Understanding Concepts

1. Spring A has a force constant of 68 N/m. Spring B has a force constant of 48 N/m. Which spring is harder to stretch?
2. If you pull northward on a spring, in what direction does the spring exert a force on you?
3. An ideal spring has a force constant of 25 N/m.
   (a) What magnitude of force would the spring exert on you if you stretched it from equilibrium by 16 cm? by 32 cm?
   (b) What magnitude of force would you have to exert on the spring to compress it from equilibrium by 16 cm? by 32 cm?
4. **Figure 5** shows the design of a tire-pressure gauge. The force constant of the spring in the gauge is 3.2×10^2 N/m. Determine the magnitude of the force applied by the air in the tire if the spring is compressed by 2.0 cm. Assume the spring is ideal.

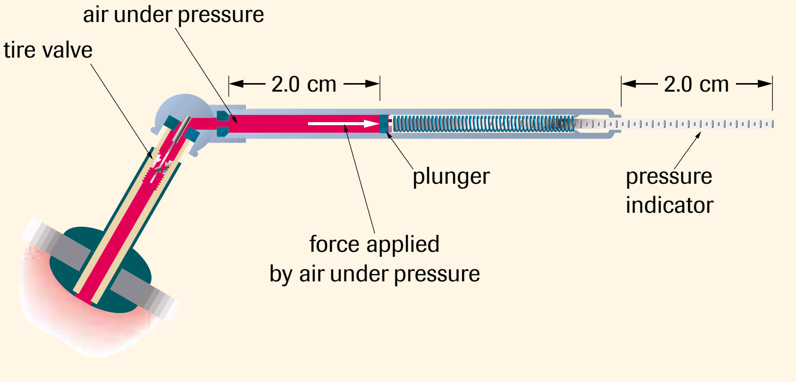


Figure 5
A pressure gauge indicates the force per unit area, a quantity measured in pascals, or newtons per square metre (1 Pa = 1 N/m^2).

Answers

3. (a) 4.0 N; 8.0 N
   (b) 4.0 N; 8.0 N
4. 6.4 N

5. A 1.37-kg fish is hung from a vertical spring scale with a force constant of 5.20×10^2 N/m. The spring obeys Hooke's law.
(a) By how much does the spring stretch if it stretches slowly to a new equilibrium position?
(b) If the fish is attached to the unstretched spring scale and allowed to fall, what is the net force on the fish when it has fallen 1.59 cm?
(c) Determine the acceleration of the fish after it has fallen 2.05 cm.

Applying Inquiry Skills

6. (a) Draw a graph of F_x as a function of x for an ideal spring, where F_x is the x-component of the force exerted *by* the spring *on* whatever is stretching (or compressing) it to position x. Include both positive and negative values of x.
(b) Is the slope of your graph positive or negative?

Making Connections

7. Spring scales are designed to measure weight but are sometimes calibrated to indicate mass. You are given a spring scale with a force constant of 80.0 N/m.
(a) Prepare a data table to indicate the stretch that would occur if masses of 1.00 kg, 2.00 kg, and on up to 8.00 kg were suspended from the scale at your location.
(b) Draw a scale diagram to show the calibration of the scale if it is set up to measure
(i) mass at your location
(ii) weight at your location
(c) If both springs in (b) were taken to the top of a high mountain, would they give the correct values? Explain.

Answers

5. (a) 0.0258 m
(b) 5.16 N [down]
(c) 2.02 m/s^2 [down]

Elastic Potential Energy

When an archer draws a bow, work is done on the limbs of the bow, giving them potential energy. The energy stored in objects that are stretched, compressed, bent, or twisted is called **elastic potential energy**. In the case of the bow, the stored energy can be transferred to the arrow, which gains kinetic energy as it leaves the bow.

elastic potential energy (E_e) the energy stored in an object that is stretched, compressed, bent, or twisted

To derive an equation for elastic potential energy, we consider the work done on an ideal spring in stretching or compressing it. Recall from Practice question 7 in Section 4.1 that the area under the line on a force-displacement graph indicates the work. For a constant force, the area is a rectangle. However, the force applied to an ideal spring depends on the displacement, so the area of the graph is a triangle (**Figure 6**). Since the area of a triangle is equal to $\frac{1}{2}bh$, we have:

$$W = \frac{1}{2}x(kx)$$

$$W = \frac{1}{2}kx^2$$

where W is the work, k is the force constant of the spring, and x is the amount of stretch or compression of the spring from the equilibrium position. Since this work has been transformed into elastic potential energy, we can rewrite the equation as

$$E_e = \frac{1}{2}kx^2$$

where E_e is the elastic potential energy.

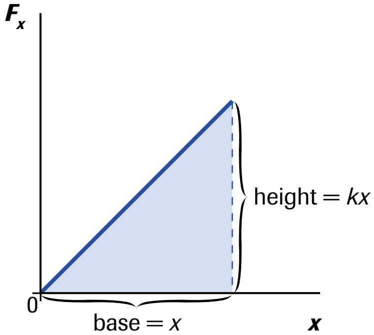


Figure 6
The magnitude of the force applied to a spring as a function of x

Elastic potential energy can be transformed into other forms of energy, such as the kinetic energy of an arrow shot by a bow, the sound energy of a guitar string, or the gravitational potential energy of a pole-vaulter at the top of the jump. As you can see from these examples, elastic potential energy can be stored in objects other than springs.

LEARNING TIP

Simplified Symbols
Various symbols are used to distinguish initial and final conditions. For example, $\vec{v}_i$ or $\vec{v}_1$ can be used to represent an initial velocity, and $\vec{v}_f$ or $\vec{v}_2$ can be used to represent a final velocity. We can also use the prime symbol (′) to represent the final condition. For example, we can use E_K for the initial kinetic energy, and E_K' for the final kinetic energy. Using the prime symbol helps simplify the equations for the law of conservation of energy and the law of conservation of momentum.

SAMPLE problem 3

An apple of mass 0.10 kg is attached to a vertical spring with a force constant of 9.6 N/m. The apple is held so that the spring is at its unstretched equilibrium position, then it is allowed to fall. Neglect the mass of the spring and its kinetic energy.

(a) How much elastic potential energy is stored in the spring when the apple has fallen 11 cm?

(b) What is the speed of the apple when it has fallen 11 cm?

Solution

(a) We measure the extension x of the spring from its original unstretched position ($x = 0$) and choose $+x$ to be downward (**Figure 7**).

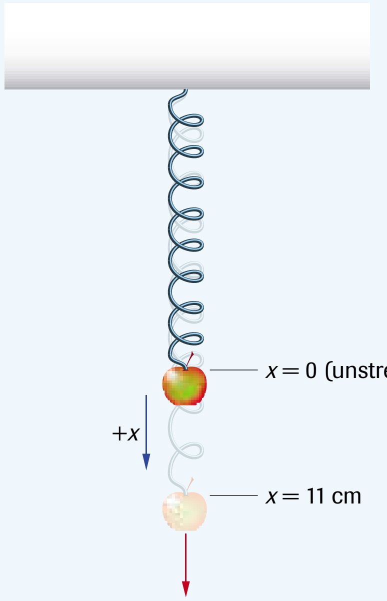


Figure 7

$x = 11\text{ cm} = 0.11\text{ m}$

$k = 9.6\text{ N/m}$

$E_e = ?$

$$E_e = \frac{1}{2}kx^2$$

$$= \frac{1}{2}(9.6\text{ N/m})(0.11\text{ m})^2$$

$$E_e = 5.8 \times 10^{-2}\text{ J}$$

The elastic potential energy stored in the spring is 5.8×10^{-2} J.

(b) We use the prime symbol (′) to represent the final condition of the apple. To apply the law of conservation of energy to determine v', we include the elastic potential energy.

$m = 0.10$ kg

$x = 0.11$ m (for the gravitational potential energy of the apple at the initial position relative to the final position)

$v = 0$

$k = 9.6$ N/m

$g = 9.8$ m/s^2

$x' = 0.11$ m (the extension of the spring when the apple is at the final position)

$E_K = E_e = 0$

$v' = ?$

$$\begin{aligned} E_T &= E_T' \\ E_g + E_K + E_e &= (E_g + E_K + E_e)' \\ E_g &= (E_K + E_e)' \\ mgx &= \frac{1}{2}mv'^2 + \frac{1}{2}kx'^2 \\ \frac{1}{2}mv'^2 &= mgx - \frac{1}{2}kx'^2 \\ v' &= \pm\sqrt{2gx - \frac{kx'^2}{m}} \\ &= \pm\sqrt{2(9.8\text{ m/s}^2)(0.11\text{ m}) - \frac{(9.6\text{ N/m})(0.11\text{ m})^2}{0.10\text{ kg}}} \\ v' &= \pm 1.0\text{ m/s} \end{aligned}$$

We choose the positive root because speed is always positive. The speed of the apple is 1.0 m/s.

SAMPLE problem 4

A group of students participating in an annual "Spring Wars Contest" is given a spring and the following challenge: Launch the spring so that it leaves a launching pad at an angle of 32.5° above the horizontal and strikes a target at the same elevation, a horizontal distance of 3.65 m away (**Figure 8**). Friction and air resistance are negligible.

(a) What measurements must the students make before they perform the calculations and launch their springs?

(b) Calculate the stretch needed for the spring to reach the target if the spring's mass is 15.4 g and its force constant is 28.5 N/m.

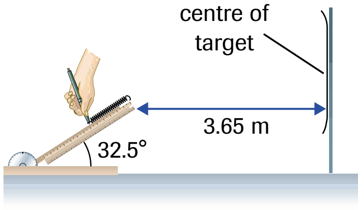


Figure 8
A student-designed launching pad for a spring

Solution

(a) As the spring is stretched, it gains elastic potential energy $E_e = \frac{1}{2}kx^2$. According to the law of conservation of energy, when the spring is released, this energy is converted into kinetic energy ($E_K = \frac{1}{2}mv^2$). The spring then moves as a projectile, covering a horizontal range given by the projectile motion equation from Section 1.4, $\Delta x = \frac{v_i^2}{g}\sin 2\theta$. (Note that Δx is the horizontal range, which is not to be confused with the extension x of a spring.) The force constant and the mass of the spring must be measured experimentally. The other variables are either given or can be calculated.

(b) We begin by calculating the speed of a projectile needed to cover the horizontal range.

$\Delta x = 3.65 \text{ m}$ $\theta = 32.5°$
$g = 9.80 \text{ m/s}^2$ $v = ?$

$$\Delta x = \frac{v^2}{g}\sin 2\theta$$
$$v^2 = \frac{g\Delta x}{\sin 2\theta}$$
$$v = \pm\sqrt{\frac{g\Delta x}{\sin 2\theta}}$$
$$= \pm\sqrt{\frac{(9.80 \text{ m/s}^2)(3.65 \text{ m})}{\sin 2(32.5°)}}$$
$$= \pm 6.28 \text{ m/s}$$
$$v = 6.28 \text{ m/s}$$

We choose the positive root because speed is always positive. Since the elastic potential energy changes into kinetic energy, we apply the law of conservation of energy equation to find the stretch x of the spring:

$m = 15.4 \text{ g} = 0.0154 \text{ kg}$ $v = 6.28 \text{ m/s}$
$k = 28.5 \text{ N/m}$ $x = ?$

$$E_e = E_K$$
$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2$$
$$x^2 = \frac{mv^2}{k}$$
$$x = \pm\sqrt{\frac{mv^2}{k}}$$
$$= \pm\sqrt{\frac{(0.0154 \text{ kg})(6.28 \text{ m/s})^2}{28.5 \text{ N/m}}}$$
$$x = \pm 0.146 \text{ m}$$

The required stretch is 0.146 m, or 14.6 cm. (The negative root would apply to a compression spring.)

TRY THIS activity — Hitting the Target

Using ideas from Sample Problem 4, design an adjustable launching pad for firing a spring (provided by your teacher) from any angle above the horizontal toward a target at least 3.00 m away and at the same vertical height as the spring. Determine the mass and force constant of your spring, and then calculate the stretch needed to launch the spring at a given angle to hit the target. If the spring is not "ideal," make adjustments to the launch so that your spring comes closer to the target.

Perform this activity away from other people. Wear safety goggles in case the spring misfires.

Practice

Understanding Concepts

8. **Figure 9** is a graph of the force as a function of stretch for a certain spring.
(a) Is the force applied *to* or *by* the spring? Explain your answer.
(b) Determine the force constant of the spring.
(c) Use the graph to determine the elastic potential energy stored in the spring after it has been stretched 35 cm.

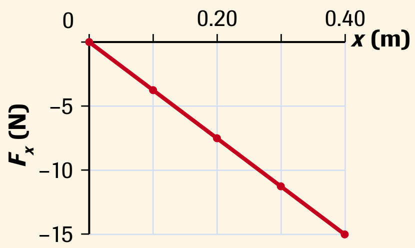


Figure 9

9. A spring has a force constant of 9.0×10^3 N/m. What is the elastic potential energy stored in the spring when it is (a) stretched 1.0 cm and (b) compressed 2.0 cm?

10. A child's toy shoots a rubber dart of mass 7.8 g, using a compressed spring with a force constant of 3.5×10^2 N/m. The spring is initially compressed 4.5 cm. All the elastic potential energy is converted into the kinetic energy of the dart.
(a) What is the elastic potential energy of the spring?
(b) What is the speed of the dart as it leaves the toy?

11. In a game, a small block is fired from a compressed spring up a plastic ramp into various holes for scoring. The mass of the block is 3.5×10^{-3} kg. The spring's force constant is 9.5 N/m. Friction is negligible.
(a) If the block is to slide up the ramp through a vertical height of 5.7 cm, by how much must the spring be compressed?
(b) If friction were not negligible, would your answer in (a) increase, decrease, or remain the same? Explain.

12. A 0.20-kg mass is hung from a vertical spring of force constant 55 N/m. When the spring is released from its unstretched equilibrium position, the mass is allowed to fall. Use the law of conservation of energy to determine
(a) the speed of the mass after it falls 1.5 cm
(b) the distance the mass will fall before reversing direction

13. A horizontal spring, of force constant 12 N/m, is mounted at the edge of a lab bench to shoot marbles at targets on the floor 93.0 cm below. A marble of mass 8.3×10^{-3} kg is shot from the spring, which is initially compressed a distance of 4.0 cm. How far does the marble travel horizontally before hitting the floor?

Applying Inquiry Skills

14. You are designing a toy that would allow your friends to bounce up and down when hanging (safely) onto a vertical spring.
(a) What measurement(s) would you make to determine the approximate force constant the spring would need to allow a maximum stretch of 75 cm when a person was suspended at rest from it?
(b) Estimate the approximate force constant for such a spring. Show your calculations.

Making Connections

15. Scientists analyze the muscles of a great variety of animals and insects. For example, when a flea jumps, the energy is provided not by muscles alone, but also by an elastic protein that has been compressed like a spring. If a flea of mass 2.0×10^2 μg jumps vertically to a height of 65 mm, and 75% of the energy comes from elastic potential energy stored in the protein, determine the initial quantity of elastic potential energy. Neglect energy losses due to air resistance.

Answers

8. (b) 38 N/m
   (c) 2.3 J
9. (a) 0.45 J
   (b) 1.8 J
10. (a) 0.35 J
    (b) 9.5 m/s
11. (a) 2.0 cm
12. (a) 0.48 m/s
    (b) 0.071 m
13. 0.66 m
15. 9.6×10^{-8} J

Simple Harmonic Motion

simple harmonic motion (SHM) periodic vibratory motion in which the force (and the acceleration) is directly proportional to the displacement

When a mass on the end of a spring vibrates in line with the central axis of the spring, it undergoes *longitudinal vibration*. Consider the longitudinal vibration of a mass on a flat surface, connected to the end of a horizontal spring that can be stretched or compressed (**Figure 10(a)**). The mass is initially at its equilibrium or rest position ($x = 0$). A force is then applied to pull the mass to a maximum displacement, called the *amplitude* A (**Figure 10(b)**). If the mass is released at this stage, the force exerted by the spring accelerates it to the left, as in **Figure 10(c)**. The force exerted by the spring varies with the stretch x according to Hooke's law, $F_x = -kx$.

After the mass in **Figure 10(c)** is released, it accelerates until it reaches maximum speed as it passes through the equilibrium position. The mass then begins to compress the spring so that the displacement is to the left. However, since the restoring force of the spring is now to the right, the acceleration is also to the right. Again, the displacement and the acceleration are in opposite directions. The mass slows down and comes to a momentary stop at $x = -A$, as shown in **Figure 10(d)**, and then moves to the right through the equilibrium position at maximum speed, and reaches $x = A$ again.

Since we are neglecting friction both in the spring and between the mass and the surface, this back-and-forth motion continues indefinitely in **simple harmonic motion** (SHM). SHM is defined as a periodic vibratory motion in which the force (and the acceleration) is directly proportional to the displacement. Be careful not to confuse simple harmonic motion with other back-and-forth motions. For example, if basketball players are running back and forth across a gym during practice, their motion is not SHM, even though the time taken for each trip may be constant.

DID YOU KNOW?

Walking and SHM

As you are walking, your foot swings back and forth with a motion that resembles the SHM of a pendulum. The speed of your foot during parts of the cycle is approximately 1.5 times the speed of your forward motion.

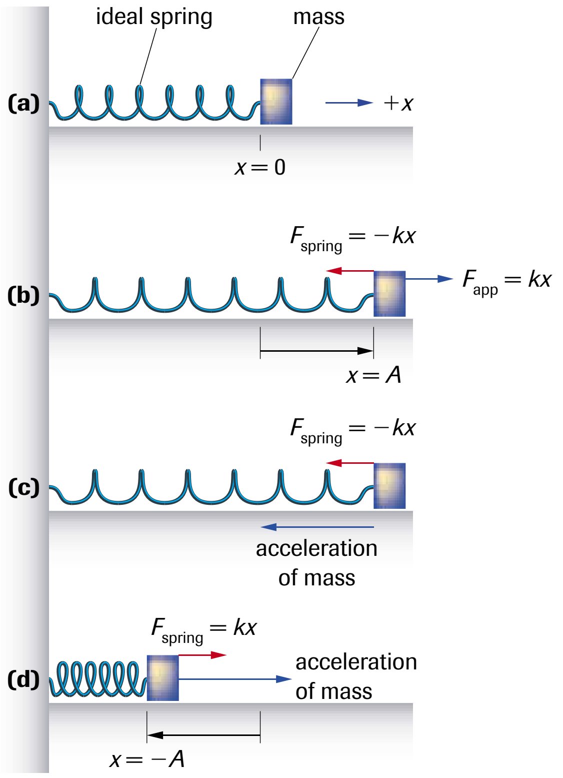


Figure 10
Using the longitudinal vibration of a mass-spring system to define simple harmonic motion (SHM)
(a) The equilibrium position
(b) The position of maximum stretch
(c) Releasing the mass
(d) The position of maximum compression

A convenient way to analyze SHM mathematically is to combine Hooke's law and Newton's second law with a *reference circle* (**Figure 11**). Imagine a mass attached to a horizontal spring, vibrating back and forth with SHM. At the same time, a handle pointing upward from a rotating disk revolves with uniform circular motion; the circular motion of the handle provides the reference circle. The frequency of revolution of the circular motion equals the frequency of vibration of the SHM, and the motions synchronize with one another. (Technically, we can say that the motions are in phase with one another.) Furthermore, the radius of the circle equals the amplitude of the SHM. A bright light source can be aimed from the side of the disk so that it casts a shadow of the upright handle onto the mass in SHM, and this shadow appears to have the same motion as the mass. This verifies that we can use equations from uniform circular motion to derive equations for SHM.

Recall that the magnitude of the acceleration of an object in uniform circular motion with a radius r and a period T is given by

$$a_c = \frac{4\pi^2 r}{T^2}$$

which we can rewrite as

$$T^2 = \frac{4\pi^2 r}{a_c} \quad \text{or} \quad T = 2\pi\sqrt{\frac{r}{a_c}}$$

Since $r = A$ for the reference circle in **Figure 11**,

$$T = 2\pi\sqrt{\frac{A}{a_c}}$$

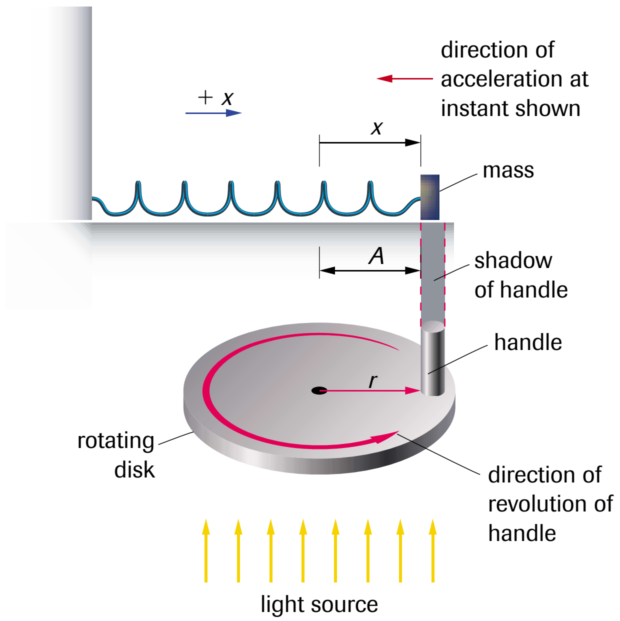


Figure 11
A reference circle. The handle on the disk is revolving with uniform circular motion, at the same frequency as the mass on the end of the spring is undergoing SHM. As the light source causes a shadow of the handle to be superimposed on the mass, the motions appear to be identical when viewed from the side. We use the motion of the reference circle to derive equations for SHM.

This period is not only the period of revolution of any point on the reference circle, but it is also the period of the mass undergoing SHM, since this mass has the same motion as the shadow of the handle.

Although the handle on the reference circle is undergoing uniform circular acceleration, its shadow is undergoing the same acceleration as the mass attached to the spring. This acceleration is not constant, as can be shown by applying Hooke's law ($F_x = -kx$) and Newton's second law ($F_x = ma_x$). If we equate the right-hand sides of these equations, then $-kx = ma_x$, from which $a_x = \frac{-kx}{m}$. Thus, since k and m are constants, the acceleration of a mass (and the shadow of the handle) undergoing SHM is proportional to the displacement, x, from the equilibrium position. Furthermore, the acceleration is opposite to the direction of the displacement, as indicated by the negative sign.

The relationship between the displacement and the acceleration can be written as $\frac{-x}{a_x} = \frac{m}{k}$; that is, the ratio of the displacement to the acceleration is constant. But in the equation that we developed for the period of SHM, the ratio $\frac{A}{a_c}$ is one specific value of the more general ratio of $\frac{-x}{a_x}$. Thus, a general equation for the period of SHM is

$$T = 2\pi\sqrt{\frac{-x}{a_x}}$$

The equation always yields a positive value under the square root sign because x and a_x have opposite signs.

If we substitute $\frac{-x}{a_x} = \frac{m}{k}$, then

$$T = 2\pi\sqrt{\frac{m}{k}}$$

where T is the period in seconds, m is the mass in kilograms, and k is the force constant of the spring in newtons per metre.

LEARNING TIP

The Period and Frequency of SHM
Like other periodic motions, SHM has a period and a frequency. The period T, measured in seconds, is the amount of time for one complete cycle. Frequency f, which is measured in hertz (Hz), is the number of cycles per second. Since period and frequency are reciprocals of one another

$f = \frac{1}{T}$ and $T = \frac{1}{f}$.

Since the frequency is the reciprocal of the period,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

These equations for the SHM of a mass–spring system apply even if the motion is vertical. The horizontal motion was used in the derivations because we did not have to consider gravity.

SAMPLE problem 5

A 0.45-kg mass is attached to a spring with a force constant of 1.4×10^2 N/m. The mass-spring system is placed horizontally, with the mass resting on a surface that has negligible friction. The mass is displaced 15 cm, and is then released. Determine the period and frequency of the SHM.

Solution

m = 0.45 kg

$k = 1.4 \times 10^2$ N/m

A = 15 cm = 0.15 m

T = ?

f = ?

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$= 2\pi\sqrt{\frac{0.45\text{ kg}}{1.4 \times 10^2\text{ N/m}}}$$

$$T = 0.36\text{ s}$$

Now, $f = \frac{1}{T}$

$$= \frac{1}{0.36\text{ s}}$$

$$f = 2.8\text{ Hz}$$

The period and the frequency of the motion are 0.36 s and 2.8 Hz.

Practice

Understanding Concepts

16. A vertical mass–spring system is bouncing up and down with SHM of amplitude A. Identify the location(s) at which
(a) the magnitude of the displacement from the equilibrium position is at a maximum
(b) the speed is at a maximum
(c) the speed is at a minimum
(d) the magnitude of the acceleration is at a maximum
(e) the magnitude of the acceleration is at a minimum

17. Determine the period and frequency, in SI units, for the following:
(a) a human eye blinks 12 times in 48 s
(b) a compact disc rotates at a rate of 210 revolutions per minute
(c) the A string on a guitar vibrates 2200 times in 5.0 s

DID YOU KNOW ?

Simple Pendulums

A simple pendulum undergoes SHM if the oscillations have a small amplitude. In this case, the equation for the period is $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the pendulum and g is the magnitude of the acceleration due to gravity.

Answers

17. (a) 4.0 s; 0.25 Hz
(b) 0.29 s; 3.5 Hz
(c) 2.3×10^{-3} s; 4.4×10^2 Hz

18. A 0.25-kg mass is attached to the end of a spring that is attached horizontally to a wall. When the mass is displaced 8.5 cm and then released, it undergoes SHM. The force constant of the spring is 1.4×10^2 N/m. The amplitude remains constant.
(a) How far does the mass move in the first five cycles?
(b) What is the period of vibration of the mass-spring system?

19. A 0.10-kg mass is attached to a spring and set into 2.5-Hz vibratory motion. What is the force constant of the spring?

20. What mass, hung from a spring of force constant 1.4×10^2 N/m, will give a mass-spring system a period of vibration of 0.85 s?

Applying Inquiry Skills

21. Show that $\sqrt{\frac{x}{a}}$ and $\sqrt{\frac{m}{k}}$ are dimensionally equivalent.

Making Connections

22. To build up the amplitude of vibration in a trampoline, you move up and down on the trampoline 6.0 times in 8.0 s without losing contact with the surface.
(a) Estimate the force constant of the trampoline.
(b) If you bounce into the air above the trampoline with a regular period of bouncing, are you undergoing SHM? Explain.

Answers

18. (a) 1.7×10^2 cm
(b) 0.27 s

19. 25 N/m

20. 2.6 kg

Energy in Simple Harmonic Motion and Damped Harmonic Motion

We have seen that the elastic potential energy in an ideal spring when it is stretched or compressed a displacement x is $\frac{1}{2}kx^2$. Let us now look at the energy transformations in an ideal spring that undergoes SHM, as in **Figure 12**. The spring is first stretched to the right, to $x = A$, from its equilibrium position, then released. The elastic potential energy is at a maximum at $x = A$:

$$E_e = \frac{1}{2}kA^2$$

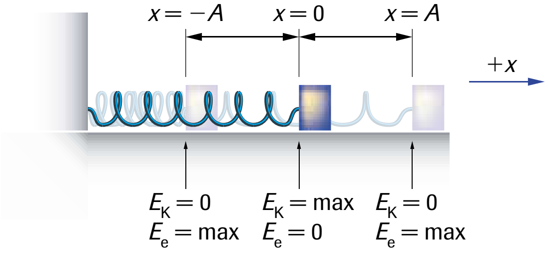


Figure 12
Mechanical energy in a mass–spring system

According to the law of conservation of energy, when the mass is released, the total energy E_T of the system is the sum of the elastic potential energy in the spring and the kinetic energy of the mass. Thus,

$$E_T = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

where k is the spring constant, x is the displacement of the mass from the equilibrium position, m is the mass at the end of the spring, and v is the instantaneous speed of the mass. As we will now see, the conservation of mechanical energy can be used to solve problems involving SHM.

SAMPLE problem 6

A 55-g box is attached to a horizontal spring of force constant 24 N/m. The spring is then compressed to a position $A = 8.6$ cm to the left of the equilibrium position. The box is released and undergoes SHM.

(a) What is the speed of the box when it is at position $x = 5.1$ cm from the equilibrium position?

(b) What is the maximum speed of the box?

Solution

(a) We use the prime symbol (′) to represent the final condition. We apply the law of conservation of mechanical energy at the two positions of the box, the initial position A and the final position x'.

$A = 8.6 \text{ cm} = 0.086 \text{ m}$

$m = 55 \text{ g} = 0.055 \text{ kg}$

$x' = 5.1 \text{ cm} = 0.051 \text{ m}$

$k = 24 \text{ N/m}$

$v' = ?$

$$E_T = E_T'$$

$$E_e + E_K = E_e' + E_K'$$

$$\frac{kA^2}{2} + 0 = \frac{kx'^2}{2} + \frac{mv'^2}{2}$$

$$kA^2 = kx'^2 + mv'^2$$

$$v' = \sqrt{\frac{k}{m}(A^2 - x'^2)} \quad \text{(discarding the negative root)}$$

$$= \sqrt{\frac{24 \text{ N/m}}{0.055 \text{ kg}}\left((0.086 \text{ m})^2 - (0.051 \text{ m})^2\right)}$$

$$v' = 1.4 \text{ m/s}$$

The speed of the box is 1.4 m/s.

(b) The maximum speed occurs when $x' = 0$.

$$v' = \sqrt{\frac{k}{m}(A^2 - x'^2)}$$

$$= \sqrt{\frac{24 \text{ N/m}}{0.055 \text{ kg}}(0.086 \text{ m})^2}$$

$$v' = 1.8 \text{ m/s}$$

The maximum speed of the box is 1.8 m/s.

damped harmonic motion periodic or repeated motion in which the amplitude of vibration and the energy decrease with time

In many practical situations involving a mass–spring system, it would be a disadvantage to have SHM. For example, if you were to step onto a bathroom scale to find your mass or weight, you would not want the spring in the scale to undergo SHM. You would expect the spring to settle down quickly and come to rest so you could observe the reading. This “settling down” is called *damping*. **Damped harmonic motion** is periodic or repeated motion in which the amplitude of vibration and thus the energy decrease with time. A typical displacement-time curve representing damped harmonic motion with a fairly long damping time is shown in **Figure 13**.

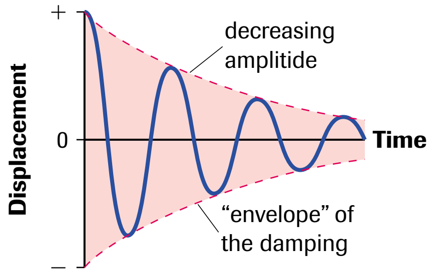


Figure 13
A displacement-time curve representing damped harmonic motion. The overall outline of the curve is called the *envelope of the damping*.

One way to study the damping properties of a real spring is to set up a mass–spring system vertically, start the mass vibrating with an appropriate amplitude, and observe the vibrations. Investigation 4.5.2, in the Lab Activities section at the end of this chapter, gives you an opportunity for such a study.

A bathroom scale is one example of a device specifically designed for damping. Another example is the system of springs and shock absorbers in a car. When a wheel goes over a bump in the road, the wheel's spring and shock absorber are compressed easily, but they are designed to quickly stop bouncing up and down. The energy given to the spring and shock absorber is dissipated, or transformed into other types of energy. To find out more about the spring and shock absorbers on automobiles, you can perform Activity 4.5.1 in the Lab Activities section at the end of this chapter.

Practice

Understanding Concepts

23. **Figures 14(a)** and **(b)** show a mass–spring system undergoing SHM at maximum compression and maximum extension.
(a) At what length(s) of the spring is the speed of the mass at a minimum? What is that speed?
(b) At what length(s) of the spring is the speed of the mass at a maximum? What is that speed?
(c) What is the amplitude of the SHM?

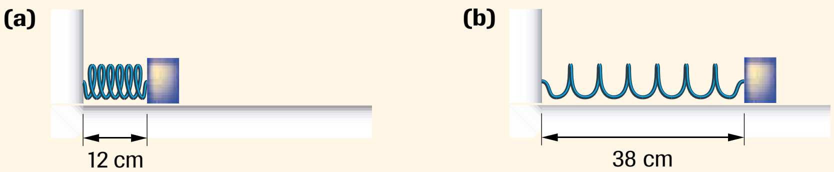


Figure 14

24. The maximum energy of a mass–spring system undergoing SHM is 5.64 J. The mass is 0.128 kg and the force constant is 244 N/m.
(a) What is the amplitude of the vibration?
(b) Use two different approaches to determine the maximum speed of the mass.
(c) Find the speed of the mass when it is 15.5 cm from the equilibrium position.

25. The amplitude of vibration of a mass on a spring experiencing SHM is 0.18 m. The mass is 58 g and the force constant is 36 N/m.
(a) Find the maximum energy of the system and the maximum speed of the mass.
(b) What amplitude of vibration would be required to double the maximum energy?
(c) What is the maximum speed of the mass at this new energy?

26. Prove that the maximum speed of a mass on a spring in SHM is given by $2\pi fA$.

INVESTIGATION 4.5.2

Analyzing Forces and Energies in a Mass–Spring System (p. 222)
If you were given a spring and appropriate measuring apparatus, you could determine the force constant of a spring. You could then apply the conservation of energy to predict the motion that will occur when a known mass, attached vertically to the end of the spring, is released from rest from the unstretched equilibrium position. What measurements would you need to make to determine the damping properties of the spring when the mass is set into vibration?

ACTIVITY 4.5.1

Achieving a Smooth and Safe Ride (p. 223)
Springs and shock absorbers on an automobile help create a smooth and safe ride for the occupants. What happens to the energy of vibration produced when the automobile goes over a bump?

Answers

23. (a) 12 cm; 38 cm; zero
(b) 25 cm; zero
(c) 13 cm

24. (a) 0.215 m
(b) 9.39 m/s
(c) 6.51 m/s

25. (a) 0.58 J; 4.5 m/s
(b) 0.25 m
(c) 6.3 m/s

Applying Inquiry Skills

27. (a) Determine the dimensions of the expression $\sqrt{\frac{k}{m}(A^2 - x^2)}$.

(b) Explain in words the meaning of this expression.

Making Connections

28. State whether each of the following devices is designed to have fast, medium, or slow damping. Give a reason for each answer.

(a) the prongs of a tuning fork

(b) the needle on an analog voltmeter

(c) a guitar string

(d) saloon doors (of the swinging type)

(e) the string on an archer's bow after the arrow leaves the bow

SUMMARY *Elastic Potential Energy and Simple Harmonic Motion*

- Hooke's law for an ideal spring states that the magnitude of the force exerted by or applied to a spring is directly proportional to the displacement the spring has moved from equilibrium.
- The constant of proportionality k in Hooke's law is the force constant of the spring, measured in newtons per metre.
- Elastic potential energy is the energy stored in objects that are stretched, compressed, twisted, or bent.
- The elastic potential energy stored in a spring is proportional to the force constant of the spring and to the square of the stretch or compression.
- Simple harmonic motion (SHM) is periodic vibratory motion such that the force (and thus the acceleration) is directly proportional to the displacement.
- A reference circle can be used to derive equations for the period and frequency of SHM.
- The law of conservation of mechanical energy can be applied to a mass–spring system and includes elastic potential energy, kinetic energy, and, in the case of vertical systems, gravitational potential energy.
- Damped harmonic motion is periodic motion in which the amplitude of vibration and the energy decrease with time.

Section 4.5 Questions

Understanding Concepts

Note: For the following questions, unless otherwise stated, assume that all springs obey Hooke's law.

1. Two students pull equally hard on a horizontal spring attached firmly to a wall. They then detach the spring from the wall and pull horizontally on its ends. If they each pull equally hard, is the amount of stretch of the spring equal to, greater than, or less than the first stretch? Explain your answer. (*Hint:* Draw an FBD for the spring in each case.)

2. Is the amount of elastic potential energy stored in a spring greater when the spring is stretched 2.0 cm than when it is compressed by the same amount? Explain your answer.

3. What does "harmonic" mean in the term "simple harmonic motion?"

4. State the relationship, if any, between the following sets of variables. Where possible, write a mathematical variation (proportionality) statement based on the appropriate equation.

(a) period and frequency

(b) acceleration and displacement in SHM

(c) period and the force constant for a mass on a spring in SHM

(d) the maximum speed of a body in SHM and the amplitude of its motion

5. A student of mass 62 kg stands on an upholstered chair containing springs, each of force constant 2.4×10^3 N/m. If the student is supported equally by six springs, what is the compression of each spring?

6. What magnitude of force will stretch a spring of force constant 78 N/m by 2.3 cm from equilibrium?

7. The coiled spring in a hand exerciser compresses by 1.85 cm when a force of 85.5 N is applied. Determine the force needed to compress the spring by 4.95 cm.

8. A trailer of mass 97 kg is connected by a spring of force constant 2.2×10^3 N/m to an SUV. By how much does the spring stretch when the SUV causes the trailer to undergo an acceleration of magnitude 0.45 m/s^2?

9. A grapefruit of mass 289 g is attached to an unstretched vertical spring of force constant 18.7 N/m, and is allowed to fall.
(a) Determine the net force and the acceleration on the grapefruit when it is 10.0 cm below the unstretched position and moving downward.
(b) Air resistance will cause the grapefruit to come to rest at some equilibrium position. How far will the spring be stretched?

10. A bungee jumper of mass 64.5 kg (including safety gear) is standing on a platform 48.0 m above a river. The length of the unstretched bungee cord is 10.1 m. The force constant of the cord is 65.5 N/m. The jumper falls from rest and just touches the water at a speed of zero. The cord acts like an ideal spring. Use conservation of energy to determine the jumper's speed at a height of 12.5 m above the water on the first fall.

11. A toy car is attached to a horizontal spring. A force of 8.6 N exerted on the car causes the spring to stretch 9.4 cm.
(a) What is the force constant of the spring?
(b) What is the maximum energy of the toy-spring system?

12. If the maximum amplitude of vibration that a human eardrum can withstand is 1.0×10^{-7} m, and if the energy stored in the eardrum membrane is 1.0×10^{-13} J, determine the force constant of the eardrum.

13. A 22-kg crate slides from rest down a ramp inclined at 29° to the horizontal (**Figure 15**) onto a spring of force constant 8.9×10^2 N/m. The spring is compressed a distance of 0.30 m before the crate stops. Determine the total distance the crate slides along the ramp. Friction is negligible.

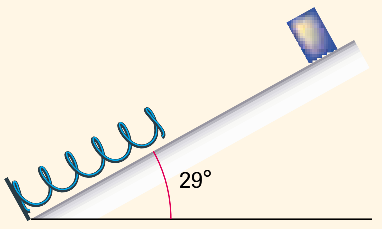


Figure 15

14. A 0.20-kg ball attached to a vertical spring of force constant 28 N/m is released from rest from the unstretched equilibrium position of the spring. Determine how far the ball falls, under negligible air resistance, before being brought to a momentary stop by the spring.

Applying Inquiry Skills

15. **Figure 16** shows the energy relationships of a 0.12-kg mass undergoing SHM on a horizontal spring. The quantity x is the displacement from the equilibrium position.
(a) Which line represents (i) the total energy, (ii) the kinetic energy, and (iii) the elastic potential energy?
(b) What is the amplitude of the SHM?
(c) What is the force constant of the spring?
(d) What is the maximum speed of the mass?

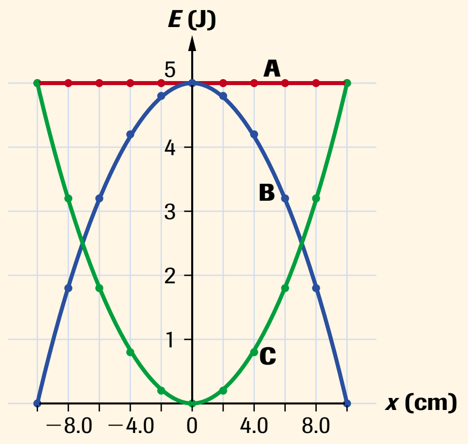


Figure 16

16. You are given a spring comprised of 24 coils that has a force constant of 24 N/m.
(a) If this spring were cut into two equal pieces, would the force constant of each new spring be equal to, greater than, or less than 24 N/m? Explain.
(b) With your teacher's permission, design and carry out an experiment to test your answer in (a). Explain what you discover.

Making Connections

17. The shock absorbers in the suspension system of a truck are in such poor condition that they have no effect on the behaviour of the springs attached to the axles. Each of the two identical springs attached to the rear axle supports 5.5×10^2 kg. After going over a severe bump, the rear end of the truck vibrates through six cycles in 3.5 s. Determine the force constant of each spring.

18. In designing components to be sent on board a satellite, engineers perform tests to ensure that the components can withstand accelerations with a magnitude as high as 25g. In one test, the computer is attached securely to a frame that is vibrated back and forth in SHM with a frequency of 8.9 Hz. What is the minimum amplitude of vibration used in this test?