# A1 Math Skills

## Significant Digits and Rounding Off Numbers

Two types of quantities are used in science: exact values and measurements. Exact values include defined quantities (e.g., 1 kg = 1000 g) and counted values (e.g., 23 students in a classroom). Measurements, however, are not exact because they always include some degree of uncertainty.

In any measurement, the *significant digits* are the digits that are known reliably, or for certain, and include the single last digit that is estimated or uncertain. Thus, if the width of a piece of paper is measured as 21.6 cm, there are three significant digits in the measurement and the last digit (6) is estimated or uncertain.

The following rules are used to determine if a digit is significant in a measurement:

- All non-zero digits are significant: 345.6 N has four significant digits.
- In a measurement with a decimal point, zeroes placed before other digits are not significant: 0.0056 m has two significant digits.
- Zeroes placed between other digits are always significant: 7003 s has four significant digits.
- Zeroes placed after other digits behind a decimal are significant: 9.100 km and 802.0 kg each has four significant digits.
- Scientific notation is used to indicate if zeroes at the end of a measurement are significant: $4.50 \times 10^7$ km has three significant digits and $4.500 \times 10^7$ km has four significant digits. The same number written as 45 000 000 km has at least two significant digits, but the total number is unknown unless the measurement is written in scientific notation. (An exception to this last statement is found if the number of significant digits can be assessed by inspection: a reading of 1250 km on a car's odometer has four significant digits.)

Measurements made in scientific experiments or given in problems are often used in calculations. In a calculation, the final answer must take into consideration the number of significant digits of each measurement, and may have to be rounded off according to the following rules:

- When adding or subtracting measured quantities, the final answer should have no more than one estimated digit; in other words, the answer should be rounded off to the least number of decimals in the original measurements.
- When multiplying or dividing measured quantities, the final answer should have the same number of significant digits as the original measurement with the least number of significant digits.

### Example

A piece of paper is 48.5 cm long, 8.44 cm wide, and 0.095 mm thick.

(a) Determine the perimeter of the piece of paper.

(b) Determine the volume of the piece of paper.

### Solution

(a) $L$ = 48.5 cm (The 5 is estimated.)

$w$ = 8.44 cm (The last 4 is estimated.)

$P$ = ?

$$P = 2L + 2w$$
$$= 2(48.5\text{ cm}) + 2(8.44\text{ cm})$$
$$P = 113.88\text{ cm}$$

Both digits after the decimal are estimated, so the answer must be rounded off to only one estimated digit. Thus, the perimeter is 113.9 cm.

(b) $h = 0.095\text{ mm} = 9.5 \times 10^{-3}\text{ cm}$ (two significant digits)

$V$ = ?

$$V = Lwh$$
$$= (48.5\text{ cm})(8.44\text{ cm})(9.5 \times 10^{-3}\text{ cm})$$
$$= 3.88873\text{ cm}^3$$
$$V = 3.9\text{ cm}^3$$

The answer is rounded off to two significant digits, which is the least number of significant digits of any of the original measurements.

Other rules must be taken into consideration in some situations. Suppose that after calculations are complete, the answer to a problem must be rounded off to three significant digits. Apply the following rules of rounding:

- If the first digit to be dropped is 4 or less, the preceding digit is not changed; for example, 8.674 is rounded to 8.67.
- If the first digit to be dropped is greater than 5, or if it is a 5 followed by at least one non-zero digit, the preceding digit is increased by 1; for example, 8.675 123 is rounded up to 8.68.

- If the first digit to be dropped is a lone 5 or a 5 followed by zeroes, the preceding digit is not changed if it is even, but is increased by 1 if it is odd; for example, 8.675 is rounded up to 8.68 and 8.665 is rounded to 8.66. (This rule exists to avoid the accumulated error that would occur if the 5 were always to round up. It is followed in this text, but not in all situations, such as in the use of your calculator or some computer software. This rule is not crucial in your success in solving problems.)

When solving multi-step problems, round-off error occurs if you use the rounded off answer from the first part of the question in subsequent parts. Thus, when doing calculations, record all the digits or store them in your calculator until the final answer is determined, and then round off the answer to the correct number of significant digits. For example, in a multi-step sample problem that involves parts (a) and (b), the answer for part (a) is written to the correct number of significant digits, but all the digits of the answer are used to solve part (b).

## Scientific Notation

Extremely large and extremely small numbers are awkward to write in common decimal notation, and do not always convey the number of significant digits of a measured quantity. It is possible to accommodate such numbers by changing the metric prefix so that the number falls between 0.1 and 1000; for example, 0.000 000 906 kg can be expressed as 0.906 mg. However, a prefix change is not always possible, either because an appropriate prefix does not exist or because it is essential to use a particular unit of measurement. In these cases, it is best to use *scientific notation*, also called standard form. Scientific notation expresses a number by writing it in the form $a \times 10^n$, where $1 \leq |a| < 10$ and the digits in the coefficient $a$ are all significant. For example, the Sun's mass is $1.99 \times 10^{30}$ kg and the period of vibration of a cesium-133 atom (used to define the second) is $1.087\ 827\ 757 \times 10^{-8}$ s.

Calculations involving very large and small numbers are simpler using scientific notation. The following rules must be applied when performing mathematical operations.

- For addition and subtraction of numbers in scientific notation:

  Change all the factors to a common factor, that is the same power of 10, and add or subtract the numbers. In general, $ax + bx = (a + b)x$.

### *Example*

$$\begin{aligned} 1.234 \times 10^5 + 4.2 \times 10^4 &= 1.234 \times 10^5 + 0.42 \times 10^5 \\ &= (1.234 + 0.42) \times 10^5 \\ &= 1.654 \times 10^5 \end{aligned}$$

This answer is rounded off to $1.65 \times 10^5$ so the answer has only one estimated digit, in this case two digits after the decimal.

- For multiplication and division of numbers in scientific notation:

  Multiply or divide the coefficients, add or subtract the exponents, and express the result in scientific notation.

### *Example*

$$\left(1.36 \times 10^4 \frac{\text{kg}}{\text{m}^3}\right)(3.76 \times 10^3\ \text{m}^3) = 5.11 \times 10^7\ \text{kg}$$

### *Example*

$$\begin{aligned} \frac{(4.51 \times 10^5\ \text{N})}{(7.89 \times 10^{-4}\ \text{m})} &= 0.572 \times 10^9\ \text{N/m} \\ &= 5.72 \times 10^8\ \text{N/m} \end{aligned}$$

When working with exponents, recall that the following rules apply:

$$\begin{aligned} x^a \cdot x^b &= x^{a+b} \\ \frac{x^a}{x^b} &= x^{a-b} \\ (x^a)^b &= x^{ab} \\ (xy)^b &= x^b y^b \\ \left(\frac{x}{y}\right)^b &= \frac{x^b}{y^b} \\ a\log x &= \log x^a \end{aligned}$$

On many calculators, scientific notation is entered using the EXP or the EE key. This key includes the "×10" from the scientific notation, so you need only enter the exponent. For example, to enter $6.51 \times 10^{-4}$, press 6.51 EXP +/−4.

## Mathematical Equations

Several mathematical equations involving geometry, algebra, and trigonometry can be applied in physics.

## Geometry

For a rectangle of length $L$ and width $w$, the perimeter $P$ and the area $A$ are

$$P = 2L + 2w$$
$$A = Lw$$

For a triangle of base $b$ and altitude $h$, the area is

$$A = \frac{1}{2}bh$$

For a circle of radius $r$, the circumference $C$ and the area are

$$C = 2\pi r$$
$$A = \pi r^2$$

For a sphere of radius $r$, the area and volume $V$ are

$$A = 4\pi r^2$$
$$V = \frac{4}{3}\pi r^3$$

For a right circular cylinder of height $h$ and radius $r$, the area and volume are

$$A = 2\pi r^2 + 2\pi rh$$
$$V = \pi r^2 h$$

## Algebra

**Quadratic formula:**

Given a quadratic equation in the form $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this equation, the discriminant $b^2 - 4ac$ indicates the number of real roots of the equation. If $b^2 - 4ac < 0$, the quadratic function has no real roots. If $b^2 - 4ac = 0$, the quadratic function has one real root. If $b^2 - 4ac > 0$, the quadratic function has two real roots.

## Trigonometry

**Trigonometric functions** for the angle $\theta$ shown in **Figure 1(a)** are

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$

**Law of Pythagoras:** For the right-angled triangle in **Figure 1(b)**, $c^2 = a^2 + b^2$, where $c$ is the hypotenuse and $a$ and $b$ are the other sides.

For the obtuse triangle in **Figure 1(c)** with angles $A$, $B$, and $C$, and opposite sides $a$, $b$, and $c$:

**Sum of the angles:** $A + B + C = 180°$

**Sine law:** $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

To use the sine law, two sides and an opposite angle (SSA) or two angles and one side (AAS) must be known.

**LEARNING TIP**

**Sine Law Caution**

The sine law can yield an acute angle rather than the correct obtuse angle when solving for an angle greater than 90°. This problem occurs because for an angle $A$ between 0° and 90°, $\sin A = \sin(A + 90°)$. To avoid this problem, always check the validity of the angle opposite the largest side of a triangle.

**Cosine law:** $c^2 = a^2 + b^2 - 2ab\cos C$

To use the cosine law, three sides (SSS), or two sides and the contained angle (SAS) must be known. Notice in the cosine law that if $C = 90°$, the equation reduces to the law of Pythagoras.

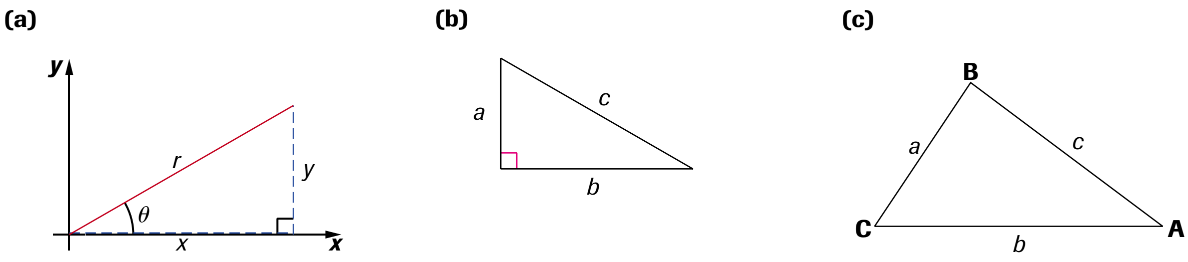


**Figure 1**
**(a)** Defining trigonometric ratios
**(b)** A right-angled triangle
**(c)** An obtuse triangle

The following trigonometric identities may be useful:

$$\cos\theta = \sin(90° - \theta)$$
$$\sin\theta = \cos(90° - \theta)$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

## Dimensional and Unit Analysis

Most physical quantities have dimensions that can be expressed in terms of five basic dimensions: mass [M], length [L], time [T], electric current [I], and temperature [$\theta$]. The SI units that correspond to these basic dimensions are kilogram [kg], metre [m], second [s], ampere [A], and kelvin [K]. The square brackets are a convention used to denote the dimension or unit of a quantity.

The process of using dimensions to analyze a problem or an equation is called *dimensional analysis*, and the corresponding process of using units is called *unit analysis*. Although this discussion focuses on dimensions only, the same process can be applied to unit analysis. Both dimensional analysis and unit analysis are tools used to determine whether an equation has been written correctly and to convert units.

### *Example*

Show that the equation for the displacement of an object undergoing constant acceleration is dimensionally correct.

$$\Delta\vec{d} = \vec{v}_i\Delta t + \frac{1}{2}\vec{a}\Delta t^2$$
$$[L] \stackrel{?}{=} \left[\frac{L}{T}\right][T] + \left[\frac{L}{T^2}\right][T^2]$$
$$[L] \stackrel{?}{=} [L] + [L]$$

The dimension of each term is the same.

Notice that in the previous example, we can ignore the number $\frac{1}{2}$ because it has no dimensions. Dimensionless quantities include:

- all plain numbers (4, $\pi$, etc.)
- counted quantities (12 people, 5 cars, etc.)
- angles (although angles have units)
- cycles
- trigonometric functions
- exponential functions
- logarithms

Derived units can be written in terms of base SI units and, thus, base dimensions. For example, the newton has base units of kg·m/s$^2$ or dimensions of [M][L][T$^{-2}$]. Can you write the dimensions of the joule and the watt? (A list of derived units is found in Appendix C.)

## Analyzing Experimental Data

Controlled physics experiments are conducted to determine the relationship between variables. The experimental data can be analyzed in a variety of ways to determine how the dependent variable depends on the independent variable(s). Often the resulting derived relationship can be expressed as an equation.

### Proportionality Statements and Graphing

The statement of how one quantity varies in relation to another is called a *proportionality statement*. (It can also be called a variation statement.) Typical proportionality statements are:

$y \propto x$ (direct proportion)

$y \propto \frac{1}{x}$ (inverse proportion)

$y \propto x^2$ (square proportion)

$y \propto \frac{1}{x^2}$ (inverse square proportion)

A proportionality statement can be converted into an equation by replacing the proportionality sign with an equal sign and including a proportionality constant. Using $k$ to represent this constant, the proportionality statements become the following equations:

$$y = kx$$
$$y = \frac{k}{x}$$
$$y = kx^2$$
$$y = \frac{k}{x^2}$$

The constant of proportionality can be determined by using graphing software or by applying regular graphing techniques as outlined in the following steps:

1. Plot a graph of the dependent variable as a function of the independent variable. If the resulting line of best fit is straight, the relationship is a direct variation. Proceed to step 3.

2. If the line of best fit is curved, replot the graph to try to get a straight line as shown in **Figure 2**. If the first replotting results in a new curved line, draw yet another graph to obtain a straight line.
3. Determine the slope and *y*-intercept of the straight line on the graph. Substitute the values into the slope/*y*-intercept form of the equation that corresponds to the variables plotted on the graph with the straight line.
4. Check the equation by substituting original data points.
5. If required, use the equation (or the straight-line graph) to give examples of interpolation and extrapolation.

## Example

Use regular graphing techniques to derive the equation relating the data given in Table 1.

**Table 1** Velocity-Time Data

| ***t* (s)** | 0.00 | 2.00 | 4.00 | 6.00 |
|---|---|---|---|---|
| **$\vec{v}$ (m/s [E])** | 10.0 | 15.0 | 20.0 | 25.0 |

## Solution

**Figure 3** is the graph that corresponds to the data in **Table 1**. The line is straight and has the following slope:

$$\text{slope} = \frac{\Delta\vec{v}}{\Delta t}$$

$$= \frac{25.0 \text{ m/s [E]} - 10.0 \text{ m/s [E]}}{6.00 \text{ s} - 0.00 \text{ s}}$$

$$\text{slope} = 2.50 \text{ m/s}^2 \text{ [E]}$$

The *y*-intercept is 10.0 m/s [E].

Using $y = mx + b$, the equation is

$$\vec{v} = 2.50 \text{ m/s}^2 \text{ [E] } (t) + 10.0 \text{ m/s [E]}$$

Verify the equation by substituting $t = 4.00$ s:

$$\vec{v} = 2.50 \text{ m/s}^2 \text{ [E] } (4.00 \text{ s}) + 10.0 \text{ m/s [E]}$$

$$= 10.0 \text{ m/s [E]} + 10.0 \text{ m/s [E]}$$

$$\vec{v} = 20.0 \text{ m/s [E]}$$

The equation is valid.

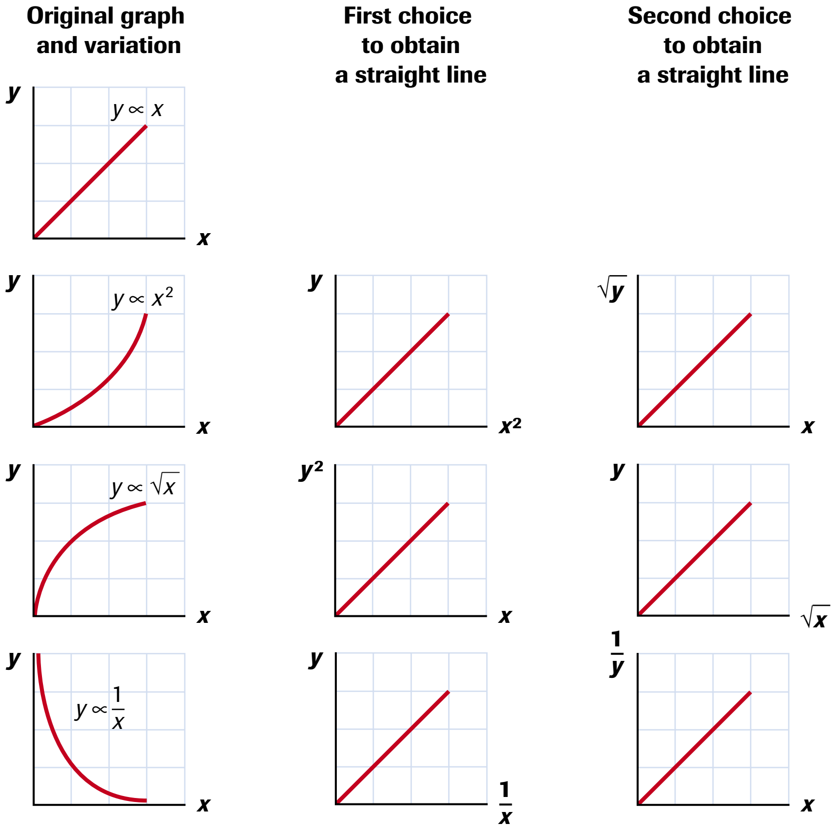


**Figure 2**
Replotting graphs to try to obtain a straight line

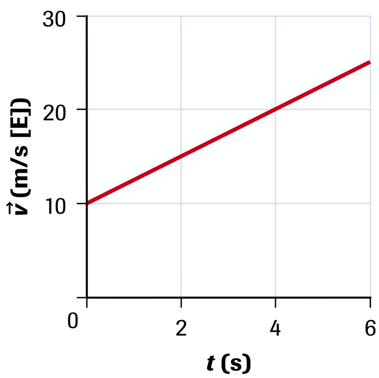


**Figure 3**
Velocity-time graph

Using $t = 3.20$ s as an example of interpolation

$$\vec{v} = 2.50 \text{ m/s}^2 \text{ [E] } (3.20 \text{ s}) + 10.0 \text{ m/s [E]}$$
$$= 8.00 \text{ m/s [E]} + 10.0 \text{ m/s [E]}$$
$$\vec{v} = 18.0 \text{ m/s [E]}$$

### *Example*

Use regular graphing techniques to derive the equation for the data in the first two rows of **Table 2**.

**Table 2** Acceleration-Mass Data

| ***m*** **(kg)** | 2.0 | 4.0 | 6.0 | 8.0 |
|---|---|---|---|---|
| $\vec{a}$ **(m/s² [E])** | 4.0 | 2.0 | 1.3 | 1.0 |
| $\frac{1}{m}$ **(kg⁻¹)*** | 0.50 | 0.25 | 0.167 | 0.125 |

* The third row is for the redrawn graph of the relationship.

**Figure 4(a)** is the graph of the data given in the first two rows of the table. **Figure 4(b)** shows the replotted graph with $m$ replaced with $\frac{1}{m}$, which produces a straight line. The slope of the straight line is

$$\text{slope} = \frac{\Delta \vec{a}}{\Delta\left(\frac{1}{m}\right)}$$

$$= \frac{3.2 \text{ m/s}^2 \text{ [E]} - 1.6 \text{ m/s}^2 \text{ [E]}}{0.40 \text{ kg}^{-1} - 0.20 \text{ kg}^{-1}}$$

$$\text{slope} = 8.0 \text{ kg}\cdot\text{m/s}^2 \text{ [E]}$$

The slope is 8.0 kg·m/s² [E], which can also be written 8.0 N [E].

The $y$-intercept is 0.0. Using $y = mx + b$, the equation is

$$\vec{a} = 8.0 \text{ kg}\cdot\text{m/s}^2 \text{ [E]} \times \frac{1}{m}$$

or $$\vec{a} = \frac{8.0 \text{ N [E]}}{m}$$

Verify the equation by substituting $m = 6.0$ kg:

$$\vec{a} = \frac{8.0 \text{ kg}\cdot\text{m/s}^2 \text{ [E]}}{6.0 \text{ kg}}$$

$$\vec{a} = 1.333 \text{ m/s}^2 \text{ [E]}$$

This value rounds off to 1.3 m/s² [E], so the equation is valid.

We can use this equation to illustrate extrapolation; for example, the acceleration when the mass is 9.6 kg the acceleration is

$$\vec{a} = \frac{8.0 \text{ kg}\cdot\text{m/s}^2 \text{ [E]}}{9.6 \text{ kg}}$$

$$\vec{a} = 0.83 \text{ m/s}^2 \text{ [E]}$$

The acceleration is 0.83 m/s² [E].

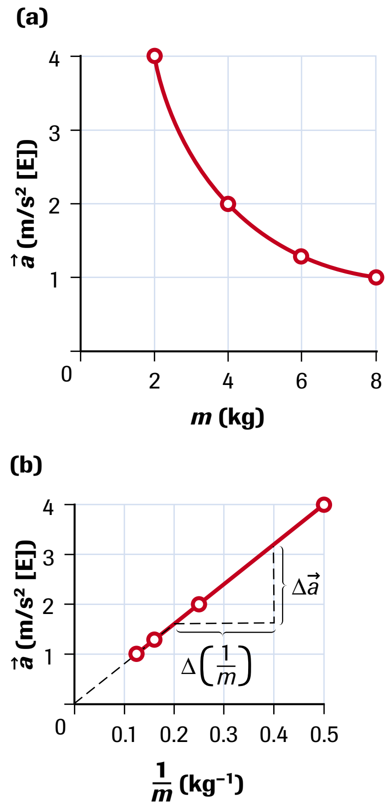


**Figure 4**
**(a)** Acceleration-mass graph
**(b)** Acceleration-$\frac{1}{\text{mass}}$ graph

## Logarithms

Many relationships in physics can be expressed as $y = kx^n$. As with other mathematical relationships, this type of equation can be analyzed using graphing software. However, another method that provides good results involves log-log graphing. If the relationship is plotted on a log-log graph, a straight line results and the equation of the line can be determined. A log-log graph has logarithmic scales on both axes. The axes can have one or more cycles, and the graph chosen depends on the domain and range of the variables to be plotted. For example, a typical log-log graph may have three cycles horizontally and two cycles vertically.

The steps in deriving an equation involving two variables using log-log graphing techniques are as follows:

1. Label the numbers on the axes of the graph starting with any power of 10, such as $10^{-3}$, $10^{-2}$, $10^{-1}$, $10^0$, $10^1$, $10^2$, $10^3$, or $10^4$. (There is no zero on a log-log graph.)

2. Plot the data on the graph and use an independent scale, such as a millimetre ruler, to determine the slope of the line. This yields the exponent $n$ in the equation $y = kx^n$.
3. Substitute data points into the equation $y = kx^n$ to determine the $k$ value, include its units, and write the final equation.
4. Check the equation by substituting original data points.

### *Example*

Use log-log graphing techniques to determine the equation for the data in **Table 3**.

**Table 3** Energy-Temperature Data

| ***T* (K)** | 2.00 | 3.00 | 4.00 |
|---|---|---|---|
| ***E* (J)** | $4.80 \times 10^3$ | $2.43 \times 10^4$ | $7.68 \times 10^4$ |

**Figure 5** shows the log-log graph of the data in **Table 3**. The slope of the line is

$$\text{slope} = \frac{\Delta y}{\Delta x}$$

$$= \frac{40\text{ mm}}{10\text{ mm}}$$

$$\text{slope} = 4$$

#### *LEARNING TIP*

**The Exponent *n* and the Constant *k***

Always round off the number found when calculating the slope of a line on a log-log graph. The slope represents the exponent, $n$, and will have values such as 1, 2, 3, 4, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc.

When two variables are involved in log-log graphing, you can find the constant $k$ by determining the $y$-intercept where $x = 1$ on the graph. However, the substitution method is usually more accurate and has the advantage of providing the units of $k$.

Using the form of the equation $y = kx^n$, where $n = 4$:

$$E = kT^4$$

To determine $k$, we use the original data:

$$k = \frac{E}{T^4}$$

$$= \frac{4.80 \times 10^3\text{ J}}{(2.00\text{ K})^4}$$

$$k = 3.00 \times 10^2\text{ J/K}^4$$

The final equation is

$$E = 3.00 \times 10^2\text{ J/K}^4(T)^4$$

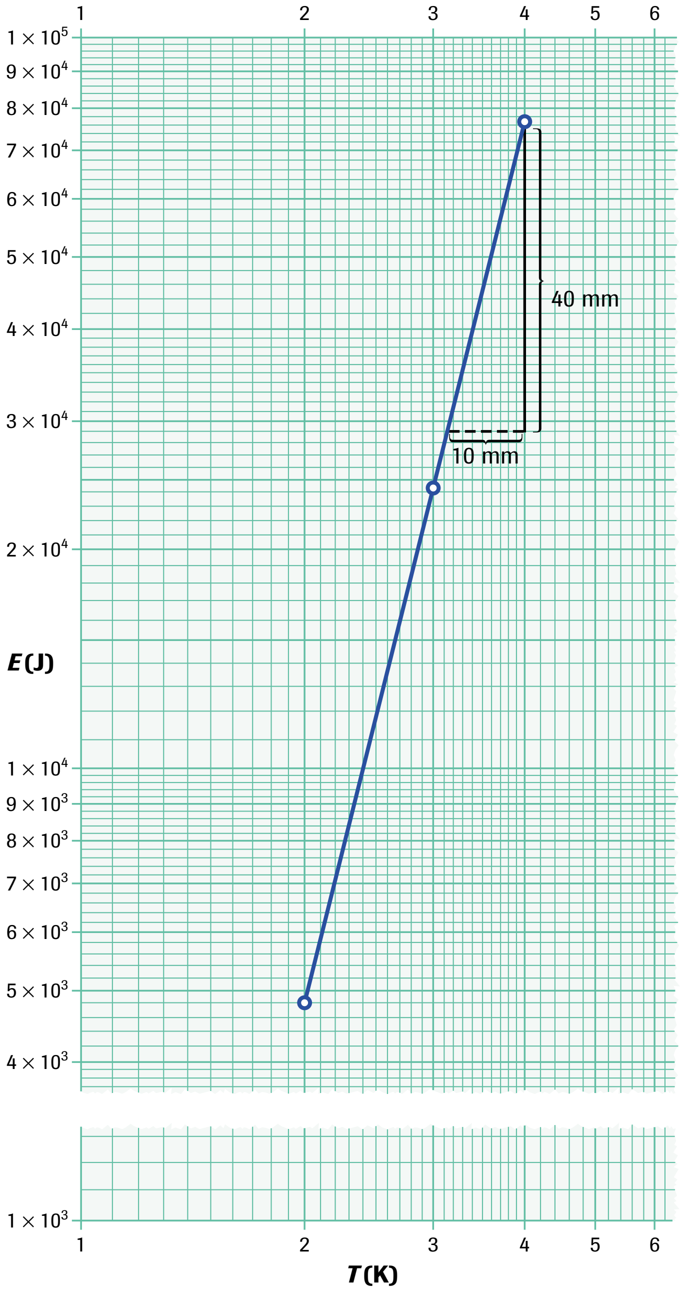


**Figure 5**
Log-log graph (with a portion removed to save space)

Check the equation using $T = 4.00$ K:

$$E = 3.00 \times 10^2\text{ J/K}^4(4.00\text{ K})^4$$
$$E = 7.68 \times 10^4\text{ J}$$

The equation is valid.

In the previous example, the values of $n$ and $k$ were calculated once. In student experimentation, the values should be calculated at least three times to improve accuracy.

The log-log graphing technique is particularly useful if three or more variables are involved in an experiment, such as in the centripetal acceleration investigation. The following steps can be applied to obtain the equation relating the variables.

1. Plot the data on log-log graph paper.
2. Find the slope $n$ of each line on the graph, and use the slopes to write the proportionality statements. For example, assume that $a$ depends on $b$ and $c$ such that the slopes are $+3$ and $-4$ respectively. The proportionality statements are $a \propto b^3$ and $a \propto c^{-4}$, or $a \propto \frac{1}{c^4}$.
3. Combine the proportionality statements $\left(\text{e.g., } a \propto \frac{b^3}{c^4}\right)$.
4. Convert the proportionality statement into an equation by inserting an equal sign and a constant $\left(\text{e.g., } a = \frac{kb^3}{c^4}\right)$.
5. Solve for the constant $k$ by taking an average of three substitutions $\left(\text{e.g., } k = \frac{ac^4}{b^3}\right)$.
6. Write the equation and include the units for $k$.
7. Check the equation by substitution.

## Error Analysis in Experimentation

In experiments involving measurement, there is always some degree of uncertainty. This uncertainty can be attributed to the instrument used, the experimental procedure, the theory related to the experiment, and/or the experimenter.

In all experiments involving measurements, the measurements and subsequent calculations should be recorded to the correct number of significant digits. However, a formal report of an experiment involving measurements should include an analysis of uncertainty, percent uncertainty, and percent error or percent difference.

*Uncertainty* is the amount by which a measurement may deviate from an average of several readings of the same measurement. This uncertainty can be estimated, so it is called the *estimated uncertainty*. Often it is assumed to be plus or minus half of the smallest division of the scale on the instrument; for example, the estimated uncertainty of 15.8 cm is $\pm 0.05$ cm or $\pm 0.5$ mm. The same applies to the *assumed uncertainty*, which is the uncertainty in a written measurement; for example, the assumed uncertainty of the Sun's mass of $1.99 \times 10^{30}$ kg is $\pm 0.005 \times 10^{30}$ kg or $\pm 5 \times 10^{27}$ kg.

Whenever calculations involving addition or subtraction are performed, the uncertainties accumulate. Thus, to find the total uncertainty, the individual uncertainties must be added. For example,

$$(34.7 \text{ cm} \pm 0.05 \text{ cm}) - (18.4 \text{ cm} \pm 0.05 \text{ cm}) = 16.3 \text{ cm} \pm 0.10 \text{ cm}$$

*Percent uncertainty* is found by dividing the uncertainty by the measured quantity and multiplying by 100%. Use your calculator to prove that 28.0 cm $\pm$ 0.05 cm has a percent uncertainty of $\pm 0.18\%$.

Whenever calculations involving multiplication or division are performed, the percent uncertainties must be added. If desired, the total percent uncertainty can be converted back to uncertainty. For example, consider the area of a certain rectangle:

$$\begin{aligned} A &= Lw \\ &= (28.0 \text{ cm} \pm 0.18\%)(21.5 \text{ cm} \pm 0.23\%) \\ &= 602 \text{ cm}^2 \pm 0.41\% \\ A &= 602 \text{ cm}^2 \pm 2.5 \text{ cm}^2 \end{aligned}$$

*Percent error* can be found only if it is possible to compare an experimental value with that of the most commonly accepted value. The equation is

$$\% \text{ error} = \frac{\text{measured value} - \text{accepted value}}{\text{accepted value}} \times 100\%$$

*Percent difference* is useful for comparing measurements when the true measurement is not known or for comparing an experimental value to a predicted value. The equation is

$$\% \text{ difference} = \frac{|\text{difference in values}|}{\text{average of values}} \times 100\%$$

*Accuracy* is a comparison of how close a measured value is to the true or accepted value. An accurate measurement has a low uncertainty.

*Precision* is an indication of the smallest unit provided by an instrument. A highly precise instrument provides several significant digits.

*Random error* occurs in measurements when the last significant digit is estimated. Random error results from variation about an average value. Such errors can be reduced by taking the average of several readings.

*Parallax* is the apparent shift in an object's position when the observer's position changes. This source of error can be reduced by looking straight at an instrument or dial.

**LEARNING TIP**

**Possible Error**
Uncertainty can also be called possible error. Thus, estimated uncertainty is estimated possible error, assumed uncertainty is assumed possible error, and percent uncertainty is percent possible error.

*Systematic error* results from a consistent problem with a measuring device or the person using it. Such errors are reduced by adding or subtracting the known error, calibrating the instrument, or performing a more complex investigation.

# Vectors

Several quantities in physics are vector quantities—quantities that have both magnitude and direction. Understanding and working with vectors is crucial in solving many physics problems.

> **LEARNING TIP**
>
> **Geometric and Cartesian Vectors**
> The vectors in this text are *geometric vectors* in one and two dimensions, which are represented as directed line segments or arrows. *Cartesian vectors* are represented as sets of ordered pairs in one and two dimensions. The properties of both geometric and Cartesian vectors can be extended to three dimensions.

## Vector Symbols

A vector is represented in a diagram by an arrow or a directed line segment. The length of the arrow is proportional to the magnitude of the vector, and the direction is the same as the direction of the vector. The tail of the arrow is the initial point, and the head of the arrow is the final point. If the vector is drawn to scale, the scale should be indicated on the diagram (**Figure 6**).

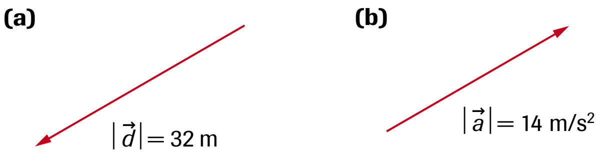


**Figure 6**
Examples of vector quantities
**(a)** Displacement vector (scale: 1 cm = 10 m)
**(b)** Acceleration vector (scale: 1 cm = 5 m/s$^2$)

In this text, a vector quantity is indicated by an arrow above the letter representing the vector (e.g., $\vec{A}, \vec{a}, \Sigma\vec{F}, \vec{p}$, etc.). The manitude of a vector is indicated either by the absolute value symbol $|\vec{A}|$ or simply $A$. The magnitude is always positive (unless it is zero).

## Directions of Vectors

The directions of vectors are indicated in square brackets following the magnitude and units of the measurement. The four compass directions east, west, north, and south are indicated as [E], [W], [N], and [S]. Other examples are [down], [forward], [11.5° below the horizontal], [toward Earth's centre], and [24° N of W] (refer to **Figure 7**).

Directions used in computers and calculators are measured counterclockwise from the $+x$-axis in an $x$-$y$ coordinate system. Using this convention, the direction [24° N of W] is simply 156°.

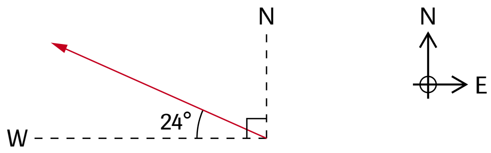


**Figure 7**
Locating the direction 24° N of W

## Multiplying a Vector by a Scalar (Scalar Multiplication)

The product of a vector and a scalar is a vector with the same direction as the original vector, but with a different magnitude (unless the scalar is 1). Thus, $8.5\vec{v}$ is a vector 8.5 times as long as $\vec{v}$ and in the same direction. Multiplying a vector by a negative scalar results in a vector in the opposite direction (**Figure 8**).

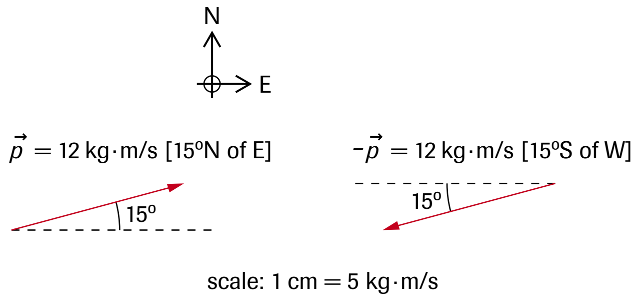


**Figure 8**
Multiplying the momentum vector $\vec{p}$ by $-1$ results in the momentum vector $-\vec{p}$.

## Components of Vectors

The *component of a vector* is the projection of a vector along an axis of a rectangular coordinate system. Any vector can be described by its rectangular components. In this text, we use two rectangular components because the situations are two-dimensional; three rectangular components are required for three-dimensional situations. Rectangular components are always perpendicular to each other, and can be called *orthogonal components*. (Orthogonal stems from the Greek word *orthos* which means "right" and *gonia* which means "angle.")

Consider the force vector $\vec{F}$ shown in **Figure 9** in which the $+x$ direction is to the left and the $+y$ direction is upward. The projection of $\vec{F}$ along the $x$-axis is $F_x$ and the projection

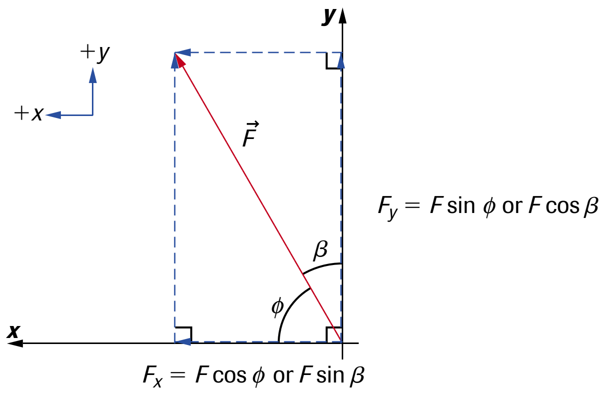


**Figure 9**
The force vector $\vec{F}$ and its components

along the $y$-axis is $F_y$. Notice that although $\vec{F}$ is a vector, its components, $F_x$ and $F_y$, are not vectors; rather they are positive or negative numbers with the same units as $\vec{F}$. In diagrams, components are often shown as broken or dashed line segments.

Notice that in **Figure 9** there are two angles that indicate each component in terms of the magnitude and direction of the vector. These angles always form a right angle (i.e., in this case $\phi + \beta = 90°$).

It is often convenient to choose a coordinate system other than a horizontal/vertical system or an east-west/north-south system. For example, consider the situation in which a skier is accelerating down a hillside inclined at an angle $\theta$ to the horizontal (**Figure 10(a)**). Solving problems related to acceleration and forces is most convenient if the $+x$ direction is the direction of the acceleration, in this case downhill; this means that the $+y$ direction must be perpendicular to the hillside. The corresponding FBD of the skier is shown in **Figure 10(b)**.

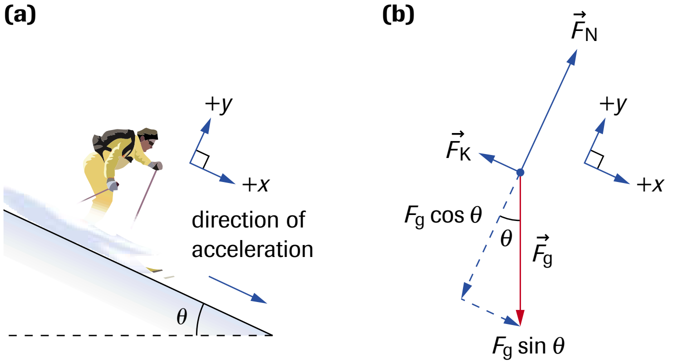


**Figure 10**
**(a)** A skier accelerating downhill
**(b)** The FBD of the skier

## Vector Addition

In arithmetic, 3 + 3 always equals 6. But if these quantities are vectors, $\vec{3} + \vec{3}$ can have any value between 0 and $\vec{6}$, depending on their orientation. Thus, vector addition must take into consideration the directions of the vectors.

To add vector quantities, the arrows representing the vectors are joined head-to-tail, with the vector representing the resultant vector joined from the tail of the first vector to the head of the last vector added (**Figure 11**). In drawing vector diagrams, the vectors can be moved around so that they are head-to-tail. When shifting a vector on a diagram, it is important that the redrawn vector have the same magnitude and direction as the original vector.

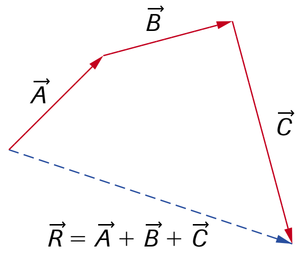


**Figure 11**
Adding three vectors

The result of adding vectors can be called the vector addition, resultant vector, resultant, net vector, or vector sum. This text uses these terms interchangeably; in diagrams, the resultant vector is a different colour or a different type of line (such as a broken line) to distinguish it from the original vectors.

Vector addition has the following properties:

- Vector addition is commutative; the order of addition does not matter: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.
- Vector addition is associative. If more than two vectors are added, it does not matter how they are grouped: $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$.

### *Example*

Use a vector scale diagram to add the following displacements and show that the addition is commutative:

$\vec{A}$ = 24 km [32°N of E]
$\vec{B}$ = 18 km [E]
$\vec{C}$ = 38 km [25°E of S]

## *Solution*

**Figure 12** shows the solution using the scale 1 cm = 10 km. The resultant displacement $\vec{R}$ is the same whether we use $\vec{R} = \vec{A} + \vec{B} + \vec{C}$ or $\vec{R} = \vec{A} + \vec{C} + \vec{B}$, thus showing that vector addition is commutative. How would you use this example to show that vector addition is associative?

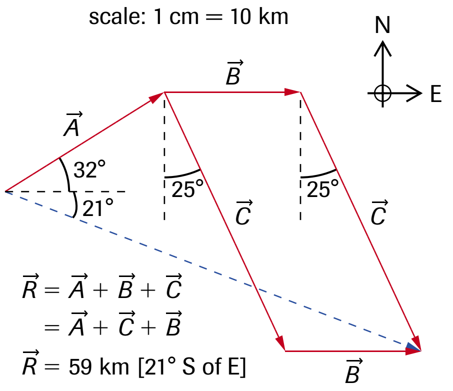


**Figure 12**
Vector addition using a vector scale diagram

The accuracy of vector addition can be improved by applying trigonometry. If two perpendicular vectors are added, the law of Pythagoras can be used to determine the magnitude of the resultant vector. A trigonometric ratio (sine, cosine, or tangent) can be used to determine the direction of the resultant vector. If the vectors are at some angle different from 90° to each other, the cosine law and the sine law can be used to determine the magnitude and direction of the resultant vector.

## *Example*

Two forces act on a single object. Determine the net force if the individual forces are

(a) $\vec{F}_1$ = 10.5 N [S] and $\vec{F}_2$ = 14.0 N [W]

(b) $\vec{F}_3$ = 10.5 N [S] and $\vec{F}_4$ = 14.0 N [25.5° W of S]

## *Solution*

(a) Refer to **Figure 13(a)**. Applying the law of Pythagoras:

$$\sum\vec{F} = \vec{F}_1 + \vec{F}_2$$

$$\left|\sum\vec{F}\right| = \sqrt{|\vec{F}_1|^2 + |\vec{F}_2|^2}$$

$$= \sqrt{(10.5\text{ N})^2 + (14.0\text{ N})^2}$$

$$\left|\sum\vec{F}\right| = 17.5\text{ N}$$

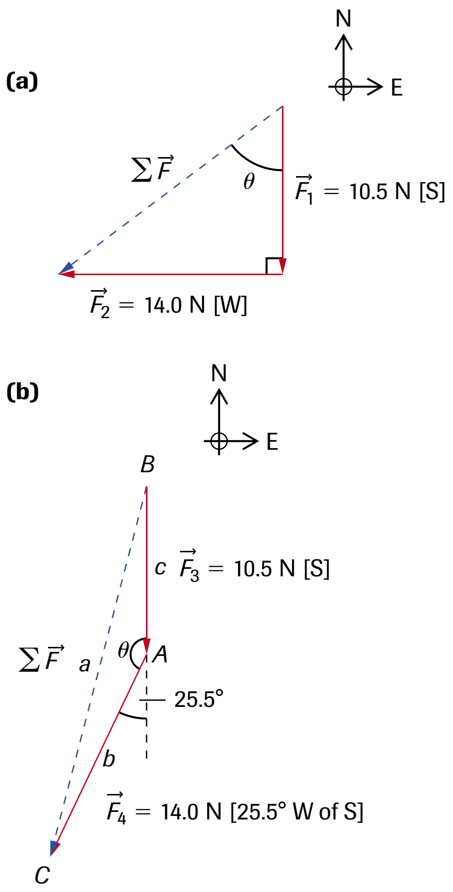


**Figure 13**
**(a)** Forces acting on the object
**(b)** Determining the net force

The angle $\theta$ is found using trigonometry:

$$\tan\theta = \frac{|\vec{F}_2|}{|\vec{F}_1|}$$

$$\theta = \tan^{-1}\frac{|\vec{F}_2|}{|\vec{F}_1|}$$

$$= \tan^{-1}\frac{14.0\text{ N}}{10.5\text{ N}}$$

$$\theta = 53.1°$$

The net force is 17.5 N [53.1° W of S].

(b) Refer to **Figure 13(b)**. Applying the cosine law:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a^2 = (14.0\text{ N})^2 + (10.5\text{ N})^2 - 2(14.0\text{ N})(10.5\text{ N})(\cos 154.5°)$$

$$a = 23.9\text{ N}$$

Applying the sine law:

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b \sin A}{a}$$

$$\sin B = \frac{(14.0\ \text{N})(\sin 154.5°)}{23.9\ \text{N}}$$

$$B = 14.6°$$

The net force is 23.9 N [14.6° W of S].

Another accurate method of vector addition is to use the components of the vectors. This method is recommended when adding three or more vectors. To add any number of vectors by components, use the following steps:

1. Define an *x*-*y* coordinate system, and indicate the $+x$ and $+y$ directions.
2. Determine the *x*- and *y*-components of all the vectors to be added.
3. Determine the net *x*-component by adding all the individual *x*-components.
4. Determine the net *y*-component by adding all the individual *y*-components.
5. Determine the magnitude and direction of the net vector by applying the law of Pythagoras and/or trigonometric ratios.

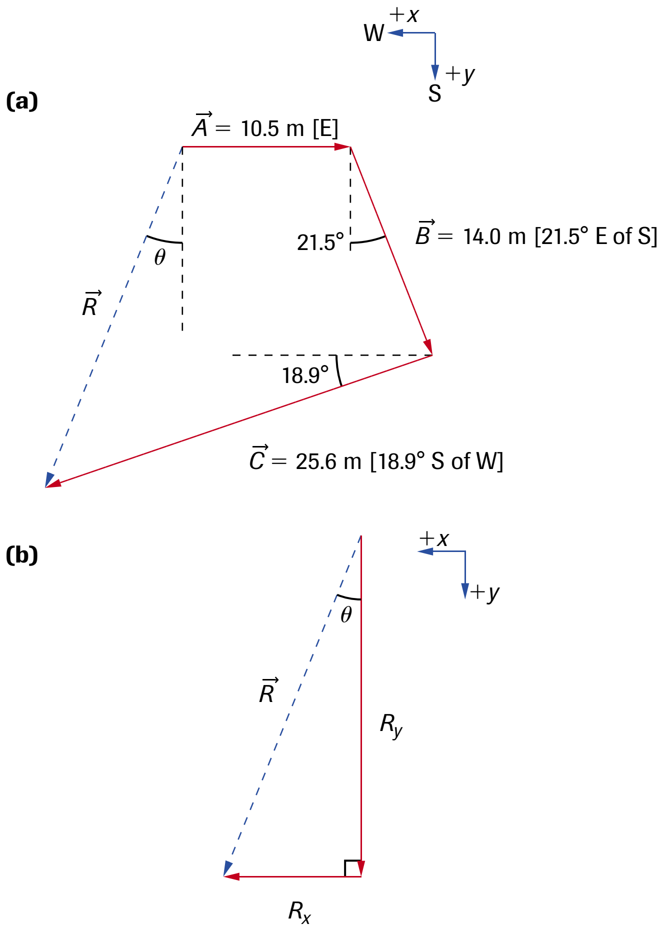


**Figure 14**
**(a)** Motion of the dog
**(b)** Determining the components of the displacements

### *Example*

Determine the resultant displacement of a dog that runs with the following displacements:

$$\vec{A} = 10.5\ \text{m [E]}$$

$$\vec{B} = 14.0\ \text{m [21.5° E of S]}$$

$$\vec{C} = 25.6\ \text{m [18.9° S of W]}$$

### *Solution*

The sketch of the motion in **Figure 14(a)** indicates that the resultant displacement is west and south of the initial position. For convenience, we choose $+x$ as west and $+y$ as south. The *x*-components of the vectors are

$$A_x = -10.5\ \text{m}$$

$$B_x = -14.0\ \text{m}\ (\sin 21.5°) = -5.13\ \text{m}$$

$$C_x = 25.6\ \text{m}\ (\cos 18.9°) = 24.2\ \text{m}$$

The *y*-components of the vectors are

$$A_y = 0\ \text{m}$$

$$B_y = 14.0\ \text{m}\ (\cos 21.5°) = 13.0\ \text{m}$$

$$C_y = 25.6\ \text{m}\ (\sin 18.9°) = 8.29\ \text{m}$$

The net *x*-component is

$$R_x = A_x + B_x + C_x$$

$$= -10.5\ \text{m} - 5.13\ \text{m} + 24.2\ \text{m}$$

$$R_x = 8.6\ \text{m}$$

The net *y*-component is

$$R_y = A_y + B_y + C_y$$

$$= 0\ \text{m} + 13.0\ \text{m} + 8.29\ \text{m}$$

$$R_y = 21.3\ \text{m}$$

As shown in **Figure 14(b)**, the magnitude of the resultant displacement is

$$R = \sqrt{R_x^2 + R_y^2}$$

$$= \sqrt{(8.6\ \text{m})^2 + (21.3\ \text{m})^2}$$

$$R = 23\ \text{m}$$

To find the direction of the resultant displacement:

$$\tan\theta = \frac{R_x}{R_y}$$

$$\theta = \tan^{-1}\frac{R_x}{R_y}$$

$$= \tan^{-1}\frac{8.6\text{ m}}{21.3\text{ m}}$$

$$\theta = 22°$$

The resultant displacement is 23 m [22° W of S].

## Vector Subtraction

The vector subtraction $\vec{A} - \vec{B}$ is defined as the vector addition of $\vec{A}$ and $-\vec{B}$, where $-\vec{B}$ has the same magnitude as $\vec{B}$, but is opposite in direction. Thus,

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}).$$

You should be able to show that $\vec{A} - \vec{B}$ does not equal $\vec{B} - \vec{A}$; in fact, the two vector subtractions are equal in magnitude, but opposite in direction.

Components can be used for vector subtraction. For example, if $\vec{C} = \vec{A} - \vec{B}$ then

$$C_x = A_x - B_x$$

and

$$C_y = A_y - B_y$$

Just as in vector addition, the subtraction of two vectors can be found using a vector scale diagram, trigonometry, or components.

### *Example*

Given that $\vec{A}$ = 35 m/s [27° N of E] and $\vec{B}$ = 47 m/s [E], determine the change in velocity $\vec{C} = \vec{A} - \vec{B}$ using a vector scale diagram and components of the vectors.

### *Solution*

The vector scale diagram in **Figure 15(a)** shows that $\vec{C} = \vec{A} + (-\vec{B})$ where the vector $-\vec{B}$ is added with the tail touching the head of $\vec{A}$. In this case, $\vec{C}$ = 23 m/s [45° W of N]. The *x*-components of the vectors are

$$A_x = -35\text{ m/s }(\cos 27°) = -31\text{ m/s}$$

$$B_x = -47\text{ m/s}$$

The *y*-components of the vectors are

$$A_y = 35\text{ m/s }(\sin 27°) = 16\text{ m/s}$$

$$B_y = 0\text{ m/s}$$

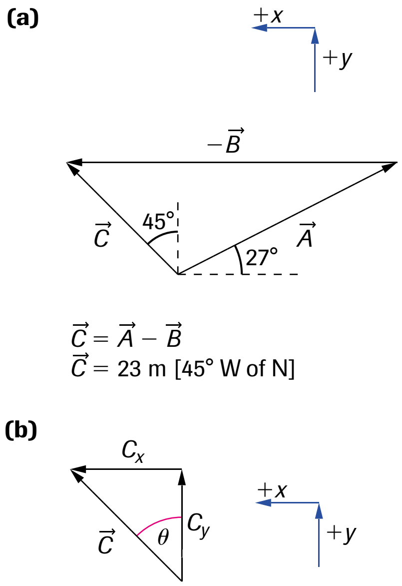


**Figure 15**
**(a)** Vector subtraction
**(b)** Determining the net change in velocity

The net *x*-component is

$$C_x = A_x - B_x$$

$$= -31\text{ m/s} - (-47\text{ m/s})$$

$$C_x = 16\text{ m/s}$$

The net *y*-component is

$$C_y = A_y - B_y$$

$$= 16\text{ m/s} - 0\text{ m/s}$$

$$C_y = 16\text{ m/s}$$

As shown in **Figure 15(b)**, the magnitude of the net change in velocity is

$$C = \sqrt{C_x^2 + C_y^2}$$

$$= \sqrt{(16\text{ m/s})^2 + 16\text{ m/s})^2}$$

$$C = 23\text{ m/s}$$

Find the direction of the net change in velocity:

$$\tan\theta = \frac{C_x}{C_y}$$

$$\theta = \tan^{-1}\frac{C_x}{C_y}$$

$$= \tan^{-1}\frac{16\text{ m}}{16\text{ m}}$$

$$\theta = 45°$$

The net change in velocity is 23 m/s [45° W of N].

## The Scalar or Dot Product of Two Vectors

The *scalar product* of two vectors is equal to the product of their magnitudes and the cosine of the angle between the vectors. The scalar product is also called the dot product because a dot can be used to represent the product symbol. An example of a scalar product is the equation for the work $W$ done by a net force $\Sigma\vec{F}$ that causes an object to move by a displacement $\Delta\vec{d}$ (Section 4.1).

$$W = \sum\vec{F}\cdot\Delta\vec{d}$$
$$= \sum F\Delta d\cos\theta$$

or

$$W = \left(\sum F\cos\theta\right)\Delta d$$

Thus, the defining equation of the scalar (or dot) product of vectors $\vec{A}$ and $\vec{B}$ is

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

where $\theta$ is the angle between $\vec{A}$ and $\vec{B}$, $A$ is the magnitude of $\vec{A}$, and $B$ is the magnitude of $\vec{B}$. Notice that $\vec{A}$ and $\vec{B}$ do not represent the same quantities.

A scalar product can be represented in a diagram as shown in **Figure 16** in which an applied force $\vec{F}_A$ is at an angle $\theta$ to the displacement of the object being pulled (with negligible friction).

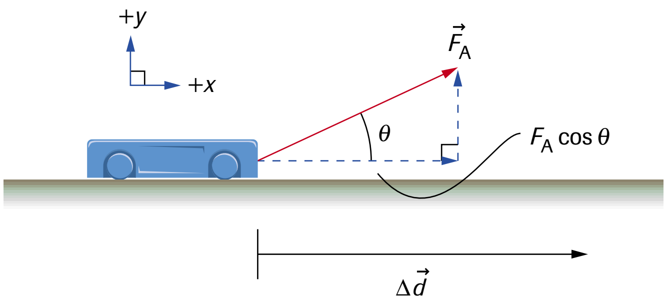


**Figure 16**
Assuming that there is negligible friction, the work done by $\vec{F}_A$ on the cart in moving it a displacement $\Delta\vec{d}$ is the scalar product, $F_A\cos\theta\Delta d$.

## The Vector or Cross Product of Two Vectors

The *vector product* of two vectors has a magnitude equal to the product of the magnitudes of the two vectors and the sine of the angle between the vectors. The vector product is also called the cross product because a "×" is used to represent the product symbol. Thus, for vectors $\vec{A}$ and $\vec{B}$, the vector product $\vec{C}$ is defined by the following equation:

$$\vec{C} = \vec{A}\times\vec{B}$$

where the magnitude is given by $C = |\vec{C}| = |AB\sin\theta|$, and the direction is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$. However, there are two distinct directions that are perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$; to determine the correct direction you can use the following rule, illustrated in **Figure 17**:

- Right-hand rule for the vector product: When the fingers of the right hand move from $\vec{A}$ *toward* $\vec{B}$, the outstretched thumb points in the direction of $\vec{C}$.

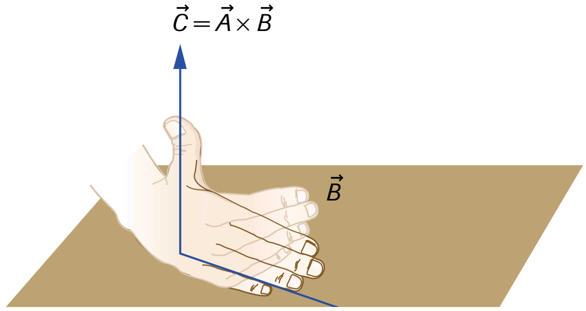


**Figure 17**
The right-hand rule to determine the direction of the vector resulting from the vector product $\vec{C} = \vec{A}\times\vec{B}$

The vector (or cross) product has the following properties:

- The order in which the vectors are multiplied matters because $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$. (Use the right-hand rule to verify this.)
- If $\vec{A}$ and $\vec{B}$ are parallel, $\theta = 0°$ or $180°$ and $\vec{A}\times\vec{B} = 0$ because $\sin 0° = \sin 180° = 0$. Thus, $\vec{A}\times\vec{A} = 0$.
- If $\vec{A}\perp\vec{B}$ ($\theta = 90°$) then $|\vec{A}\times\vec{B}| = AB$ because $\sin 90° = 1$.
- The vector product obeys the distributive law; i.e., $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$.

**LEARNING TIP**

**Alternative Notation**

In advanced physics textbooks, vectors are often written using boldface rather than with an arrow above the quantity. Thus, you may find the dot product and the cross product written as follows:

$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$
$\mathbf{A}\times\mathbf{B} = AB\sin\theta$

# Using Graphing Calculators or Computer Programs

You can use a graphing calculator or a computer-graphing program for several purposes, including finding the roots of an equation or analyzing linear functions, quadratic functions, trigonometric functions, and conic functions. You can also create a graph of given or measured data, and determine the equation relating the variables plotted or solve two simultaneous equations with two unknowns.

## Graphing Calculators

### *Example*

A ball is tossed vertically upward with an initial speed of 9.0 m/s. At what times after its release will the ball pass a position 3.0 m above the position where it is released? (Neglect air resistance.)

### *Solution*

Note: The solution given here is for the TI-83 Plus calculator. If you have a different computing calculator, refer to its instruction manual for detailed information about solving equations.

Defining upward as the positive direction and using magnitudes only, the given quantities are $\Delta d = 3.0$ m, $v_i = 9.0$ m/s, and $a = -9.8$ m/s$^2$. The constant acceleration equation for displacement is

$$\Delta d = v_i \Delta t + \frac{1}{2} a \Delta t^2$$

$$3.0 = 9.0\Delta t - 4.9\Delta t^2$$

$$4.9\Delta t^2 - 9.0\Delta t + 3.0 = 0$$

To solve for $\Delta t$, we can use the quadratic formula and enter the data into the calculator. The equation is in the form $Ax^2 + Bx + C = 0$, where $A = 4.9$, $B = -9.0$, and $C = 3.0$.

1. Store the coefficients A and B, and the constant C in the calculator:
   - 4.9 [STO▸] [ALPHA] [A]
   - [ALPHA] [:]
   - −9 [STO▸] [ALPHA] [B]
   - [ALPHA] [:]
   - 3 [STO▸] [ALPHA] [C]
   - [ENTER]
2. Enter the expression for the quadratic formula:

   $$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}:$$

   - [(] [−] [ALPHA] [B] + [2nd] [√] [(] [ALPHA] [B] [$x^2$] [−] 4 [ALPHA] [A] [ALPHA] [C] [)] [)] [÷] [(] 2 [ALPHA] [A] [)]
3. Press [ENTER] to find one solution to the time. To find the other solution, the negative must be used in front of the discriminant. The answers are 1.4 s and 0.44 s.

### *Example*

Graph the function $y = \cos x$ for $0° \le x \le 360°$.

### *Solution*

1. Put the calculator in degree mode:
   - [MODE] → "Degree" → [ENTER].
2. Enter $y = \cos x$ into the equation editor:
   - Y = [COS] [X,T,Θ,n] [)].
3. Adjust the window so that it corresponds to the given domain:
   - [WINDOW] → $X_{min} = 0$, $X_{max} = 360$, $X_{scl} = 90$ (for an interval of 90° on the $x$-axis), $Y_{min} = -1$, and $Y_{max} = 1$.
4. Graph the function using the ZoomFit:
   - [ZOOM] [0].

Consider an ellipse, which is important in physics because it is the shape of the orbits of planets and satellites. The standard form of the equation of an ellipse where the centre is the origin and the major axis is along the $x$-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a > b.$$

The vertexes of the ellipse are at $(a, 0)$ and $(-a, 0)$, as shown in **Figure 18**.

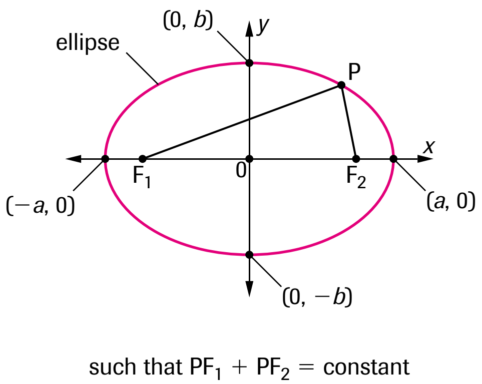


**Figure 18**
An ellipse

### *Example*

Use the "Zap-a-Graph" feature to plot an ellipse centred on the origin of an $x$-$y$ graph and determine the effect of changing the parameters of the ellipse.

### *Solution*

1. Choose ellipse from the Zap-a-Graph menu:
   - [DEFINE] → Ellipse.
2. Enter the parameters of the ellipse (e.g., $a = 6$ and $b = 4$), then plot the graph.
3. Alter the ellipse by choosing Scale from the Grid menu and entering different values.

## Graphing on a Spreadsheet

A *spreadsheet* is a computer program that can be used to create a table of data and a graph of the data. It is composed of cells indicated by a column letter (A, B, C, etc.) and a row number (1, 2, 3, etc.); thus, B1 and C3 are examples of cells (**Figure 19**). Each cell can hold a number, a label, or an equation.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | A1 | B1 | C1 | | |
| 2 | A2 | B2 | | | |
| 3 | A3 | | | | |

**Figure 19**
Spreadsheet cells

To create a data table and plot the corresponding graph, you can follow the steps outlined in the next example.

### *Example*

For an initial velocity of 5.0 m/s [E] and a constant acceleration of 4.0 m/s$^2$ [E], set up a spreadsheet for the relationship $\Delta\vec{d} = \vec{v}_i\Delta t + \frac{1}{2}\vec{a}\Delta t^2$. Plot a graph of the data from $t = 0$ s to $t = 8.0$ s at intervals of 1.0 s.

### *Solution*

1. Access the spreadsheet and label cell A1 the independent variable, in this case *t*, and cell B1 the dependent variable, in this case $\Delta d$.
2. Enter the values of *t* from 0 to 8.0 in cells A2 to A10. In cell B2 enter the right side of the equation in the following form: $v_i{*}t + \frac{1}{2}{*}a{*}t{*}t$ where "*" represents multiplication.
3. Use the cursor to select B2 down to B10 and choose the Fill Down command or right drag cell B1 down to B10 to copy the equation to each cell.
4. Command the program to graph the values in the data table (e.g., choose Make Chart, depending on the program).

A spreadsheet can be used to solve a system of two simultaneous equations, which can occur, for example, when analyzing elastic collisions (discussed in Section 5.3). In a one-dimensional elastic collision, the two equations involved simultaneously are

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$$ from the law of conservation of momentum

and $$\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2$$ from the law of conservation of energy

If the known quantities are $m_1$, $m_2$, $v_1$, and $v_2$, then the unknown quantities are $v_1'$ and $v_2'$. To find the solution to these unknowns, rewrite the equations so that one unknown is isolated and written in terms of all the other variables. In this example, enter $v_2'$ in cell A1, enter the first $v_1'$ based on the equation for the law of conservation of momentum in cell B1, and enter the second $v_1'$ (call it $v_1''$) based on the equation for the law of conservation of energy in cell C1. Then proceed to enter the data according to the previous example, using reasonable values for the variables. Plot the data on a graph and determine the intersection of the two resulting lines. That intersection is the solution to the two simultaneous equations.

# A2 Planning an Investigation

In our attempts to further our understanding of the natural world, we encounter questions, mysteries, or events that are not readily explainable. To develop explanations, we investigate using scientific inquiry. The methods used in scientific inquiry depend, to a large degree, on the purpose of the inquiry.

## Controlled Experiments

A controlled experiment is an example of scientific inquiry in which an independent variable is purposefully and steadily changed to determine its effect on a second dependent variable. All other variables are controlled or kept constant. Controlled experiments are performed when the purpose of the inquiry is to create, test, or use a scientific concept.

The common components of controlled experiments are outlined below. *Even though the presentation is linear, there are normally many cycles through the steps during an actual experiment.*

### Stating the Purpose

Every investigation in science has a purpose; for example,

- to develop a scientific concept (a theory, law, generalization, or definition);
- to test a scientific concept;
- to determine a scientific constant; or
- to test an experimental design, a procedure, or a skill.

Determine which of these is the purpose of your investigation. Indicate your decision in a statement of the purpose.

### Asking the Question

Your question forms the basis for your investigation: the investigation is designed to answer the question. Controlled experiments are about relationships, so the question could be about the effects on variable A when variable B is changed.

### Predicting/Hypothesizing

A prediction is a tentative answer to the question you are investigating. In the prediction you state what outcome you expect from your experiment.

A hypothesis is a tentative explanation. To be scientific, a hypothesis must be testable. Hypotheses can range in certainty from an educated guess to a concept that is widely accepted in the scientific community.

### Designing the Investigation

The design of a controlled experiment identifies how you plan to manipulate the independent variable, measure the response of the dependent variable, and control all the other variables in pursuit of an answer to your question. It is a summary of your plan for the experiment.

### Gathering, Recording, and Organizing Observations

There are many ways to gather and record observations during your investigation. It is helpful to plan ahead and think about what data you will need to answer the question and how best to record them. This helps to clarify your thinking about the question posed at the beginning, the variables, the number of trials, the procedure, the materials, and your skills. It will also help you organize your evidence for easier analysis.

### Analyzing the Observations

After thoroughly analyzing your observations, you may have sufficient and appropriate evidence to enable you to answer the question posed at the beginning of the investigation.

### Evaluating the Evidence and the Prediction/Hypothesis

At this stage of the investigation, you evaluate the processes that you followed to plan and perform the investigation.

You will also evaluate the outcome of the investigation, which involves evaluating any prediction you made, and the hypothesis or more established concept ("authority") the prediction was based on. You must identify and take into account any sources of error and uncertainty in your measurements.

Finally, compare the answer you predicted with the answer generated by analyzing the evidence. Is your hypothesis, or the authority, acceptable or not?

### Reporting on the Investigation

In preparing your report, your objectives should be to describe your planning process and procedure clearly and in sufficient detail that the reader could repeat the experiment exactly as you performed it, and to report your observations, your analysis, and your evaluation of your experiment accurately and honestly.

# A3 Decision Making

Modern life is filled with environmental and social issues that have scientific and technological dimensions. An issue is defined as a problem that has at least two possible solutions rather than a single answer. There can be many positions, generally determined by the values that an individual or a society holds, on a single issue. Which solution is "best" is a matter of opinion; ideally, the solution that is implemented is the one that is most appropriate for society as a whole.

The common processes involved in the decision-making process are outlined below. *Even though the sequence is presented as linear, you may go through several cycles before deciding you are ready to defend a decision.*

## Defining the Issue

The first step in understanding an issue is to explain why it is an issue, describe the problems associated with the issue, and identify the individuals or groups, called stakeholders, involved in the issue. You could brainstorm the following questions to research the issue: Who? What? Where? When? Why? How? Develop background information on the issue by clarifying facts and concepts, and identifying relevant attributes, features, or characteristics of the problem.

## Identifying Alternatives/Positions

Examine the issue and think of as many alternative solutions as you can. At this point it does not matter if the solutions seem unrealistic. To analyze the alternatives, you should examine the issue from a variety of perspectives. Stakeholders may bring different viewpoints to an issue and these may influence their position on the issue. Brainstorm or hypothesize how different stakeholders would feel about your alternatives. Perspectives that stakeholders may adopt while approaching an issue are listed in **Table 1**.

## Researching the Issue

Formulate a research question that helps to limit, narrow, or define the issue. Then develop a plan to identify and find reliable and relevant sources of information. Outline the stages of your information search: gathering, sorting, evaluating, selecting, and integrating relevant information. You may consider using a flow chart, concept map, or other graphic organizer to outline the stages of your information search. Gather information from many sources, including newspapers, magazines, scientific journals, the Internet, and the library.

## Analyzing the Issue

In this stage, you will analyze the issue in an attempt to clarify where you stand. First, you should establish criteria for evaluating your information to determine its relevance and significance. You can then evaluate your sources, determine what assumptions may have been made, and assess whether you have enough information to make your decision.

There are five steps that must be completed to effectively analyze the issue:

1. Establish criteria for determining the relevance and significance of the data you have gathered.
2. Evaluate the sources of information.

**Table 1** Some Possible Perspectives on an Issue

| | |
|---|---|
| **Cultural** | focused on customs and practices of a particular group |
| **Environmental** | focused on effects on natural processes and other living things |
| **Economic** | focused on the production, distribution, and consumption of wealth |
| **Educational** | focused on the effects on learning |
| **Emotional** | focused on feelings and emotions |
| **Aesthetic** | focused on what is artistic, tasteful, beautiful |
| **Moral/Ethical** | focused on what is good/bad, right/wrong |
| **Legal** | focused on rights and responsibilities |
| **Spiritual** | focused on the effects on personal beliefs |
| **Political** | focused on the aims of an identifiable group or party |
| **Scientific** | focused on logic or the results of relevant inquiry |
| **Social** | focused on effects on human relationships, the community |
| **Technological** | focused on the use of machines and processes |

3. Identify and determine what assumptions have been made. Challenge unsupported evidence.
4. Determine any causal, sequential, or structural relationships associated with the issue.
5. Evaluate the alternative solutions, possibly by conducting a risk-benefit analysis.

## Defending the Decision

After analyzing your information, you can answer your research question and take an informed position on the issue. You should be able to defend your preferred solution in an appropriate format—debate, class discussion, speech, position paper, multimedia presentation (e.g., computer slide show), brochure, poster, video …

Your position on the issue must be justified using the supporting information that you have discovered in your research and tested in your analysis. You should be able to defend your position to people with different perspectives. In preparing for your defence, ask yourself the following questions:

- Do I have supporting evidence from a variety of sources?
- Can I state my position clearly?
- Do I have solid arguments (with solid evidence) supporting my position?
- Have I considered arguments against my position, and identified their faults?
- Have I analyzed the strong and weak points of each perspective?

## Evaluating the Process

The final phase of decision making includes evaluating the decision the group reached, the process used to reach the decision, and the part you played in decision making. After a decision has been reached, carefully examine the thinking that led to the decision. Some questions to guide your evaluation follow:

- What was my initial perspective on the issue? How has my perspective changed since I first began to explore the issue?
- How did we make our decision? What process did we use? What steps did we follow?
- In what ways does our decision resolve the issue?
- What are the likely short- and long-term effects of our decision?
- To what extent am I satisfied with our decision?
- What reasons would I give to explain our decision?
- If we had to make this decision again, what would I do differently?

## A Risk–Benefit Analysis Model

Risk–benefit analysis is a tool used to organize and analyze information gathered in research. A thorough analysis of the risks and benefits associated with each alternative solution can help you decide on the best alternative.

- Research as many aspects of the proposal as possible. Look at it from different perspectives.
- Collect as much evidence as you can, including reasonable projections of likely outcomes if the proposal is adopted.
- Classify every individual potential result as being either a benefit or a risk.
- Quantify the size of the potential benefit or risk (perhaps as a dollar figure, or a number of lives affected, or in severity on a scale of 1 to 5).
- Estimate the probability (percentage) of that event occurring.
- By multiplying the size of a benefit (or risk) by the probability of its happening, you can assign a significance value for each potential result.
- Total the significance values of all the potential risks, and all the potential benefits and compare the sums to help you decide whether to accept the proposed action.

Note that although you should try to be objective in your assessment, your beliefs will have an effect on the outcome—two people, even if using the same information and the same tools, could come to a different conclusion about the balance of risk and benefit for any proposed solution to an issue.

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# A4 Technological Problem Solving

There is a difference between scientific and technological processes. The goal of science is to understand the natural world. The goal of technological problem solving is to develop or revise a product or a process in response to a human need. The product or process must fulfill its function but, in contrast with scientific problem solving, it is not essential to understand why or how it works. Technological solutions are evaluated based on such criteria as simplicity, reliability, efficiency, cost, and ecological and political ramifications.

*Although the sequence below is linear, there are normally many cycles through the steps in any problem-solving attempt.*

## Defining the Problem

This process involves recognizing and identifying the need for a technological solution. You need to clearly state both the question(s) that you want to investigate and the criteria you will use as guidelines to solve the problem and to evaluate your solution. In any design, some criteria may be more important than others. For example, if the product solution measures accurately and is economical, but is not safe, then it is clearly unacceptable.

## Identifying Possible Solutions

Use your knowledge and experience to propose possible solutions. Creativity is also important in suggesting novel solutions.

You should generate as many ideas as possible about the functioning of your solution and about potential designs. During brainstorming, the goal is to generate many ideas without judging them. They can be evaluated and accepted or rejected later.

To visualize the possible solutions it is helpful to draw sketches. Sketches are often better than verbal descriptions to communicate an idea.

## Planning

Planning is the heart of the entire process. Your plan will outline your processes, identify potential sources of information and materials, define your resource parameters, and establish evaluation criteria.

Seven types of resources are generally used in developing technological solutions to problems—people, information, materials, tools, energy, capital, and time.

## Constructing/Testing Solutions

In this phase, you will construct and test your prototype using systematic trial and error. Try to manipulate only one variable at a time. Use failures to inform the decisions you make before your next trial. You may also complete a cost-benefit analysis on the prototype.

To help you decide on the best solution, you can rate each potential solution on each of the design criteria using a five-point rating scale, with 1 being poor, 2 fair, 3 good, 4 very good, and 5 excellent. You can then compare your proposed solutions by totalling the scores.

Once you have made the choice among the possible solutions, you need to produce and test a prototype. While making the prototype you may need to experiment with the characteristics of different components. A model, on a smaller scale, might help you decide whether the product will be functional. The test of your prototype should answer three basic questions:

- Does the prototype solve the problem?
- Does it satisfy the design criteria?
- Are there any unanticipated problems with the design?

If these questions cannot be answered satisfactorily, you may have to modify the design or select another solution.

## Presenting the Preferred Solution

In presenting your solution, you will communicate your solution, identify potential applications, and put your solution to use.

Once the prototype has been produced and tested, the best presentation of the solution is a demonstration of its use—a test under actual conditions. This demonstration can also serve as a further test of the design. Any feedback should be considered for future redesign. Remember that no solution should be considered the absolute final solution.

## Evaluating the Solution and Process

The technological problem-solving process is cyclical. At this stage, evaluating your solution and the process you used to arrive at your solution may lead to a revision of the solution.

Evaluation is not restricted to the final step, however, it is important to evaluate the final product using the criteria established earlier, and to evaluate the processes used while arriving at the solution. Consider the following questions:

- To what degree does the final product meet the design criteria?
- Did you have to make any compromises in the design? If so, are there ways to minimize the effects of the compromises?
- Are there other possible solutions that deserve future consideration?
- Did you exceed any of the resource parameters?
- How did your group work as a team?

# A5 Lab Reports

When carrying out investigations, it is important that scientists keep records of their plans and results, and share their findings. In order to have their investigations repeated (replicated) and accepted by the scientific community, scientists generally share their work by publishing papers in which details of their design, materials, procedure, evidence, analysis, and evaluation are given.

Lab reports are prepared after an investigation is completed. To ensure that you can accurately describe the investigation, it is important to keep thorough and accurate records of your activities as you carry out the investigation.

Investigators use a similar format in their final reports or lab books, although the headings and order may vary. Your lab book or report should reflect the type of scientific inquiry that you used in the investigation and should be based on the following headings, as appropriate.

## Title

At the beginning of your report, write the number and title of your investigation. In this course the title is usually given, but if you are designing your own investigation, create a title that suggests what the investigation is about. Include the date the investigation was conducted and the names of all lab partners (if you worked as a team).

## Purpose

State the purpose of the investigation. Why are you doing this investigation?

## Question

This is the question that you attempted to answer in the investigation. If it is appropriate to do so, state the question in terms of independent and dependent variables.

## Prediction/Hypothesis

A prediction is a tentative answer to the question you are investigating. In the prediction you state what outcome you expect from your experiment.

A hypothesis is a tentative explanation. To be scientific, a hypothesis must be testable. Hypotheses can range in certainty from an educated guess to a concept that is widely accepted in the scientific community. Depending on the nature of your investigation, you may or may not have a hypothesis or a prediction.

## Experimental Design

If you designed your own investigation, this is a brief general overview (one to three sentences) of what was done. If your investigation involved independent, dependent, and controlled variables, list them. Identify any control or control group that was used in the investigation.

## Materials

This is a detailed list of all materials used, including sizes and quantities where appropriate. Be sure to include safety equipment and any special precautions when using the equipment or performing the investigation. Draw a diagram to show any complicated setup of apparatus.

## Procedure

Describe, in detailed, numbered steps, the procedure you followed in carrying out your investigation. Include steps to clean up and dispose of waste materials.

## Observations

This includes all qualitative and quantitative observations that you made. Be as precise as appropriate when describing quantitative observations, include any unexpected observations, and present your information in a form that is easily understood. If you have only a few observations, this could be a list; for controlled experiments and for many observations, a table will be more appropriate.

## Analysis

Interpret your observations and present the evidence in the form of tables, graphs, or illustrations, each with a title. Include any calculations, the results of which can be shown in a table. Make statements about any patterns or trends you observed. Conclude the analysis with a statement based only on the evidence you have gathered, answering the question that initiated the investigation.

## Evaluation

The evaluation is your judgment about the quality of evidence obtained and about the validity of the prediction and hypothesis (if present). This section can be divided into two parts:

- Did your observations provide reliable and valid evidence to enable you to answer the question? Are you confident enough in the evidence to use it to evaluate any prediction and/or hypothesis you made?
- Was the prediction you made before the investigation supported or falsified by the evidence? Based on your evaluation of the evidence and prediction, is your hypothesis or the authority you used to make your prediction supported, or should it be rejected?

The leading questions that follow should help you through the process of evaluation.

### Evaluation of the Experiment

1. Were you able to answer the question using the chosen experimental design? Are there any obvious flaws in the design? What alternative designs (better or worse) are available? As far as you know, is this design the best available in terms of controls, efficiency, and cost? How great is your confidence in the chosen design?

   You may sum up your conclusions about the design in a statement like: "The experimental design [name or describe in a few words] is judged to be adequate/inadequate because … "
2. Were the steps that you used in the laboratory correctly sequenced, and adequate to gather sufficient evidence? What improvements could be made to the procedure? What steps, if not done correctly, would have significantly affected the results?

   Sum up your conclusions about the procedure in a statement like: "The procedure is judged to be adequate/inadequate because … "
3. Which specialized skills, if any, might have the greatest effect on the experimental results? Was the evidence from repeated trials reasonably similar? Can the measurements be made more precise?

   Sum up your conclusions: "The technological skills are judged to be adequate/inadequate because … "
4. You should now be ready to sum up your evaluation of the experiment. Do you have enough confidence in your experimental results to proceed with your evaluation of the authority being tested? Based on uncertainties and errors you have identified in the course of your evaluation, what would be an acceptable percent difference for this experiment (1%, 5%, or 10%)?

   State your confidence level in a summary statement: "Based upon my evaluation of the experiment, I am not certain/I am moderately certain/I am very certain of my experimental results. The major sources of uncertainty or error are … "

### Evaluation of the Prediction and Authority

1. Calculate the percent difference for your experiment.

   $$\%\ \text{difference} = \frac{|\text{difference in values}|}{\text{average of values}} \times 100\%$$

   How does the percent difference compare with your estimated total uncertainty (i.e., is the percent difference greater or smaller than the difference you've judged acceptable for this experiment)? Does the predicted answer clearly agree with the experimental answer in your analysis? Can the percent difference be accounted for by the sources of uncertainty listed earlier in the evaluation?

   Sum up your evalution of the prediction: "The prediction is judged to be verified/inconclusive/falsified because … "
2. If the prediction was verified, the hypothesis or the authority behind it is supported by the experiment. If the results of the experiment were inconclusive or the prediction was falsified, then doubt is cast upon the hypothesis or authority. How confident do you feel about any judgment you can make based on the experiment? Is there a need for a new or revised hypothesis, or to restrict, reverse, or replace the authority being tested?

   Sum up your evaluation of the authority: "[The hypothesis or authority] being tested is judged to be acceptable/unacceptable because … "

## Synthesis

You can synthesize your knowledge and understanding in the following ways:

- Relate what you discovered in the experiment to theories and concepts studied previously.
- Apply your observations and conclusions to practical situations.