7.4 Working with Rational Exponents

YOU WILL NEED

- graph paper

GOAL

Determine the meaning of a power with a rational exponent, and evaluate expressions containing such powers.

THE SEATTLE POST-INTELLIGENCER.

LATEST NEWS FROM THE KLONDIKE.

9 O'CLOCK EDITION.

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Sixty-Eight Rich Men on the Steamer Portland.

STACKS OF YELLOW METAL!

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Special Tug Chartered by the Post-Intelligencer to Get the News.

KLONDYKE

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Store Open Until 9 O'clock Tonight.

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Clover Cream Makes the Skin Elastic.

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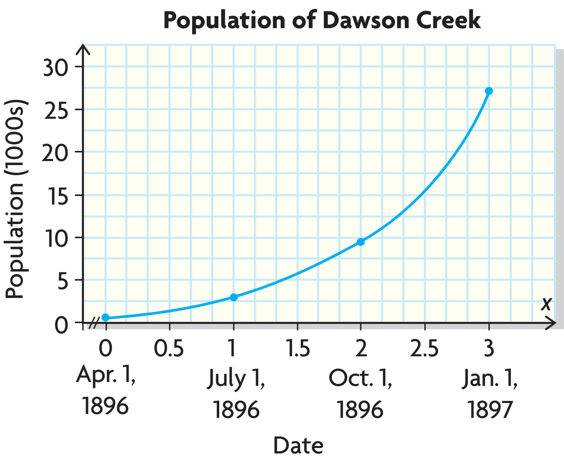
INVESTIGATE the Math

On August 16, 1896, gold was discovered near Dawson, in the Yukon region of Canada.

The population of Dawson City experienced rapid growth during this time. The population was approximately 1000 in April and 3000 in July (and grew to 30 000 at one point). The population is given in the table and is also shown on the graph.



Date	x	Population (1000s)
Apr. 1, 1896	0	1, or 3^0
July 1, 1896	1	3, or 3^1
Oct. 1, 1896	2	9, or 3^2
Jan. 1, 1897	3	27, or 3^3

? About how many people were in Dawson City in mid-May and mid-August of 1896?

A. What x-value can be used to represent the middle of May? Explain how you know.

B. Use the graph to estimate the population in the middle of May.

C. What power can be used to represent the population in the middle of May? Multiply this power by itself. What do you notice?

D. Use the exponent button on your calculator to calculate $3^{\frac{1}{2}}$. How does this value relate to your estimate in part B?

E. Use the graph to estimate the population at the beginning of May (when $x = \frac{1}{3}$). Calculate $3^{\frac{1}{3}}$ and compare it with your estimate.

F. Calculate $3^{\frac{1}{3}} \times 3^{\frac{1}{3}} \times 3^{\frac{1}{3}}$. What do you notice?

G. What is another way to express $3^{\frac{1}{3}}$ using a **radical?**

radical

the indicated root of a quantity; for example, $\sqrt[3]{8} = 2$ since $2 \times 2 \times 2 = 2^3 = 8$

H. Write a power of 3 that would estimate the population in the middle of August. Evaluate the power with your calculator and check your answer by using the graph shown.

Reflecting

I. Explain why $x^{\frac{1}{2}}$ is equivalent to $\sqrt{x}$.

J. Make a conjecture about powers with the exponent $\frac{1}{n}$.

K. Do the rules for multiplying powers with the same base still apply if the exponents are rational? Use the numbers in your table to investigate.

APPLY the Math

EXAMPLE 1 Representing a power with a positive rational exponent

Evaluate $65^{\frac{1}{3}}$.

Elaine's Solution

$65^{\frac{1}{3}} = \sqrt[3]{65}$

I know that the exponent $\frac{1}{3}$ is a way of representing the 3rd root of the base. I wrote it in radical notation.

$\sqrt[3]{65} \doteq 4$

The answer is a number whose cube is 65.

I know that $4 \times 4 \times 4 = 64$.
The answer must be close to 4.

I used my calculator to get a more accurate answer.

When an exponent is written as a decimal, it is often easier to evaluate the power if the exponent is written as its equivalent fraction.

EXAMPLE 2 Representing a power with a decimal rational exponent

Evaluate $32^{0.2}$.

Tosh's Solution

$32^{0.2} = 32^{\frac{1}{5}}$

I rewrote the power changing the exponent from 0.2 to its equivalent fraction.

$$0.2 = \frac{2}{10} = \frac{1}{5}$$

$= \sqrt[5]{32}$

I know that an exponent $\frac{1}{5}$ is a way of representing the 5th root of the number. I wrote it in radical notation.

$= 2$

Since $2^5 = 32$, the 5th root of 32 must be 2.

I entered the power into my calculator to verify my answer.

If a power involves a negative rational exponent, then you can write the exponent as the product of a fraction and an integer.

example 3 Representing a power with a negative rational exponent

Evaluate $(-27)^{-\frac{2}{3}}$.

George's Solution: Using $\frac{1}{3}$ and -2 as Exponents

$$(-27)^{-\frac{2}{3}} = \left((-27)^{\frac{1}{3}}\right)^{-2}$$

I separated the exponent into two parts, since the exponent $-\frac{2}{3}$ can be written as a product: $\frac{1}{3} \times (-2)$.

$$= \frac{1}{\left((-27)^{\frac{1}{3}}\right)^2}$$

I expressed $\left((-27)^{\frac{1}{3}}\right)^{-2}$ as a rational number, using 1 as the numerator and $\left((-27)^{\frac{1}{3}}\right)^2$ as the denominator.

$$= \frac{1}{\left((-27)^{\frac{1}{3}} \times (-27)^{\frac{1}{3}}\right)}$$

$$= \frac{1}{\left(\sqrt[3]{-27} \times \sqrt[3]{-27}\right)}$$

I determined the cube root of -27. I know that $(-3) \times (-3) \times (-3) = -27$.

$$= \frac{1}{(-3 \times -3)}$$

$$= \frac{1}{9}$$

Nadia's Solution: Using $-\frac{1}{3}$ and 2 as Exponents

$$(-27)^{-\frac{2}{3}} = \left((-27)^{-\frac{1}{3}}\right)^{-2}$$

I separated the exponent into two parts. The exponent $-\frac{2}{3}$ can be written as a product: $-\frac{1}{3} \times 2$

$$= \left(\frac{1}{(-27)^{\frac{1}{3}}}\right)^2$$

$$= \left(\frac{1}{\sqrt[3]{-27}}\right)^2$$

I expressed $(-27)^{\frac{1}{3}}$ as a radical.

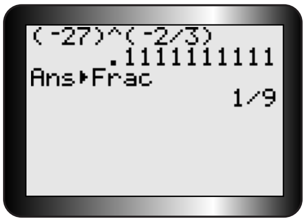
$$= \left(\frac{1}{-3}\right)^2$$

I evaluated the root and squared the result.

$$= \frac{1}{9}$$

Anjali's Solution: Using a Calculator

$(-27)^{-\frac{2}{3}}$



I used my calculator to determine the answer and then changed it from a decimal to a fraction.

The strategies you have used to evaluate numerical expressions involving integer exponents also apply when exponents are rational.

EXAMPLE 4 Selecting a strategy to evaluate an expression involving rational exponents

Simplify, then evaluate $\dfrac{(8^{\frac{1}{6}})^7}{8^{\frac{1}{2}}8^{\frac{1}{3}}}$.

Aaron's Solution

$$\frac{(8^{\frac{1}{6}})^7}{8^{\frac{1}{2}}8^{\frac{1}{3}}} = \frac{8^{\frac{7}{6}}}{8^{\frac{5}{6}}}$$

The numerator is a power of a power, so I multiplied the exponents: $\frac{1}{6} \times 7 = \frac{7}{6}$

The denominator is a product of powers, so I added the exponents: $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

To divide powers (numerator by denominator), I subtracted the exponents.

$$= 8^{\frac{7}{6} - \frac{5}{6}}$$

$$= 8^{\frac{2}{6}}$$

$$= 8^{\frac{1}{3}}$$

I wrote the fraction in lowest terms.

$$= \sqrt[3]{8}$$

$$= 2$$

I found the cube root of 8.

In Summary

Key Ideas

- A power with a rational exponent is equivalent to a radical. The rational exponent $\frac{1}{n}$ indicates the nth root of the base. If $n > 1$ and $n \in \mathbf{N}$, then $b^{\frac{1}{n}} = \sqrt[n]{b}$, where $b \neq 0$.
- If $m \neq 1$ and if m and n are both positive integers, then $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$, where $b \neq 0$.

Need to Know

- The exponent laws that apply to powers with integer exponents also apply to powers with rational exponents.
- The power button on a scientific calculator can be used to evaluate rational exponents.
- Some roots of negative numbers cannot be determined. For example, -16 does not have a real-number square root, since $(-4)^2 = (-4) \times (-4) = +16$. Odd roots can have negative bases, but even ones cannot.
- Since radicals can be written as powers with rational exponents:
  - Their products are equivalent to the products of powers. This means that $\sqrt{a} \times \sqrt{b} \times \sqrt{c} = \sqrt{a \times b \times c}$, because $a^{\frac{1}{2}} \times b^{\frac{1}{2}} \times c^{\frac{1}{2}} = (abc)^{\frac{1}{2}}$, where a, b, and $c > 0$.
  - Their quotients are equivalent to the quotient of powers. This means that $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, because $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \left(\frac{a}{b}\right)^{\frac{1}{2}}$, where a, b, and $c > 0$.

CHECK Your Understanding

1. Write in radical form. Then evaluate without using a calculator.

a) $49^{\frac{1}{2}}$ **c)** $81^{\frac{1}{4}}$ **e)** $16^{0.25}$

b) $(-125)^{\frac{1}{3}}$ **d)** $100^{\frac{1}{2}}$ **f)** $-(144)^{0.5}$

2. Write in exponent form. Then evaluate.

a) $\sqrt[10]{1024}$ **c)** $\sqrt[3]{27^4}$ **e)** $\sqrt[4]{16}$

b) $\sqrt[5]{1024}$ **d)** $\left(\sqrt[3]{-216}\right)^5$ **f)** $(\sqrt{25})^{-1}$

3. Use your calculator to evaluate each expression to the nearest hundredth.

a) $6^{\frac{2}{5}}$ **c)** $\sqrt[15]{4421}$ **e)** $10^{\frac{-2}{3}}$

b) $0.0625^{\frac{1}{4}}$ **d)** $144^{0.25}$ **f)** $200^{-0.4}$

PRACTICE

4. Evaluate each expression. Use the fact that $2^5 = 32$ and $3^4 = 81$.
K

a) $-32^{\frac{1}{5}}$
b) $\sqrt[5]{-32}$
c) $\sqrt[10]{(-32)^2}$
d) $(-2)^5$
e) -81^{-025}
f) $\sqrt[4]{81}$

5. The volume of a cube is 0.027cm^3. What does $\sqrt[3]{0.027}$ represent in the situation?
A

6. Write each expression as a single power with a positive exponent.

a) $7^{\frac{1}{2}} \times 7^2$
b) $3^4 \div 3^{\frac{1}{2}}$
c) $(16^{\frac{2}{3}})^6$
d) $\dfrac{12^{\frac{2}{3}}}{12^{\frac{1}{6}}}$
e) $(10^{\frac{5}{8}})^{-2}$
f) $2^{0.5} \times 2^2 \times 2^{2.5}$

7. Write each expression as a single power with a positive exponent.

a) $3^{-\frac{1}{2}} \div 3^{\frac{3}{2}}$
b) $(5^{-\frac{2}{5}})^{10}$
c) $8^{\frac{5}{2}} \div 8^{-\frac{5}{2}}$
d) $4^{0.3} \div 4^{0.8} \times 4^{-0.7}$
e) $9^{0.5} \times 3^0$
f) $2^3 \times 4^{-2} \div 8$

8. Given that $10^{0.5} \doteq 3.16$, determine the value of $10^{1.5}$ and $10^{-0.5}$. Explain your reasoning.

9. Write each expression using powers, then simplify. Evaluate your simplified expression.

a) $\dfrac{\sqrt{200}}{\sqrt{2}}$
b) $\sqrt{3}\sqrt{6}\sqrt{2}$
c) $\dfrac{\sqrt{98}}{\sqrt{2}}$
d) $\dfrac{\sqrt{12}}{\sqrt{8}\sqrt{6}}$
e) $\dfrac{\sqrt{15}\sqrt{10}}{\sqrt{6}}$
f) $3\sqrt{12}\sqrt{3}$

10. Write each expression as a single power with positive exponents.

a) $\dfrac{6^{\frac{3}{2}} \times 6^5}{6}$
b) $\dfrac{(8^2)^{\frac{1}{2}}}{8(8^2)}$
c) $\dfrac{12^{-\frac{3}{4}}}{12^{\frac{-1}{2}}}$
d) $\dfrac{(11^{-\frac{3}{4}})(11^{\frac{5}{8}})}{11^{\frac{3}{2}}}$
e) $4^{\frac{2}{3}} \div 4^{\frac{-1}{2}} \times 4^{\frac{5}{6}}$
f) $\left(\left(16^{-\frac{1}{2}}\right)^2\right)^{-\frac{1}{4}}$

11. Simplify. Express final answers in radical form.

a) $4^{-\frac{3}{8}}(4^2)$
b) $\dfrac{9^{\frac{4}{3}}}{9^{\frac{7}{10}}}$
c) $10^{\frac{9}{4}}(10^{-2})$
d) $(8^{\frac{1}{5}})(8^{-\frac{2}{15}})$
e) $5^{\frac{1}{2}} \times 5^{-1}$
f) $4^3 \div 4^{\frac{3}{4}}$

12. Rewrite each of the following expressions using a rational exponent. Then evaluate using your calculator. Express answers to the nearest thousandth.

a) $\sqrt[3]{120}$
b) $25^{0.75}$
c) $\sqrt[5]{13^{-2}}$
d) $10^{-0.8}$
e) $(\sqrt[3]{216})^{-2}$
f) $(\sqrt[3]{-15})^{-2}$

13. Evaluate $-(8)^{\frac{1}{3}}$ and $-(4)^{\frac{1}{2}}$ using your calculator. Compare the results.

14. Simplify. Write each answer with positive exponents.

a) $m\left(m^{\frac{2}{3}}\right)$ **c)** $\left(c^{3}\right)^{\frac{5}{6}}$ **e)** $\dfrac{s\left(s^{0.25}\right)}{\left(s^{1.5}\right)^{0.3}}$

b) $\dfrac{x}{x^{-\frac{4}{3}}}$ **d)** $\left(b^{\frac{8}{9}}\right)^{\frac{9}{4}}$ **f)** $m\left(m^{\frac{2}{3}}\right)m^{-\frac{5}{3}}$

15. Simplify. Write each answer with positive exponents.

a) $\dfrac{t\left(t^{-\frac{8}{5}}\right)}{t^{\frac{3}{5}}}$ **c)** $\dfrac{\left(y^{5}\right)^{-\frac{9}{5}}}{\left(y^{-\frac{3}{2}}\right)^{4}}$ **e)** $\left(x^{\frac{1}{3}} \div x^{\frac{2}{3}}\right)^{-3}$

b) $\dfrac{\left(x^{\frac{7}{4}}\right)^{\frac{1}{2}}}{x^{-\frac{5}{6}}}$ **d)** $\left(\dfrac{a^{\frac{1}{2}}a^{\frac{3}{2}}}{\left(a^{-2}\right)^{\frac{1}{2}}}\right)$ **f)** $\left(\left(b^{-8}\right)^{\frac{-1}{2}}\right)^{\frac{-3}{4}}$

16. Evaluate $64^{-\frac{5}{3}}$ without a calculator. Explain each of the steps in your evaluation.

17. T Determine the value of the variable that makes each of the following true. Express each answer to the nearest hundredth.

a) $1.05 = \sqrt[3]{M}$ **c)** $N^{\frac{1}{5}} - 3 = 0$ **e)** $x^{\frac{2}{3}} = 4$

b) $2.5 = \sqrt[4]{T}$ **d)** $\dfrac{x^{5}}{x^{2}} = 125$ **f)** $y^{-0.25} = \dfrac{1}{3}$

18. Write in exponential form. Use the exponent laws to simplify and then evaluate.

a) $\sqrt{1000} \times \sqrt[3]{1000} \div \sqrt[6]{1000}$ **b)** $\dfrac{\left(\sqrt{64}\right)^{2}}{\sqrt[3]{64}}$

19. C Use your knowledge of exponents to express $32^{\frac{4}{5}}$ in two other ways. Which one is easier to evaluate if you do not use a calculator?

Extending

20. Simplify.

a) $\dfrac{4 + 4^{-1}}{4 - 4^{-1}}$ **b)** $\dfrac{5^{-2} - 5^{-1}}{5^{-2} + 5^{-1}}$ **c)** $\dfrac{\sqrt{4^{3}}\left(\sqrt[5]{4^{4}}\right)}{\sqrt{2^{10}}}$

21. Write as a radical: $\left(21^{6}\right)^{-\frac{1}{4}}$

22. If $a = 2$ and $b = -1$, which expression has the greater value?

a) $\dfrac{a^{-2b}a^{-b+2}}{\left(a^{-2}\right)^{b}}$ **b)** $\dfrac{\left(a^{b}\right)^{-3}a^{-1(-2b)}}{\left(a^{-b}\right)^{3}}$