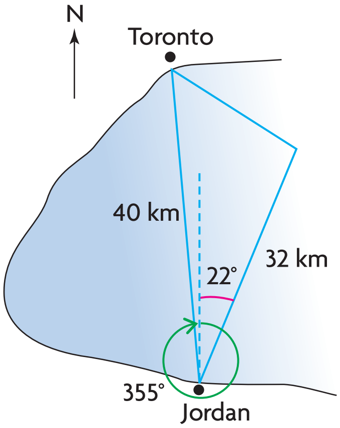
5.5 Solving Problems by Using Acute-Triangle Models

GOAL

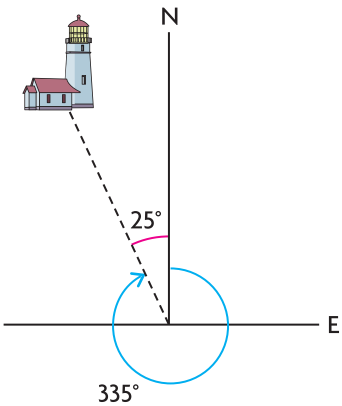
Solve problems involving the primary trigonometric ratios and the sine and cosine laws.

LEARN ABOUT the Math



Steve leaves the marina at Jordan on a 40 km sailboat race across Lake Ontario intending to travel on a bearing of 355°, but an early morning fog settles. By the time it clears, Steve has travelled 32 km on a bearing of 22°.

bearing
the direction in which you have to move in order to reach an object. A bearing is a clockwise angle from magnetic north. For example, the bearing of the lighthouse shown is 335°.

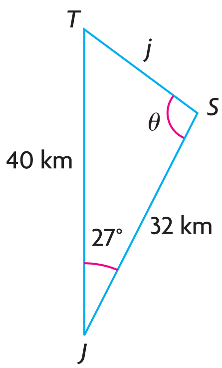


? In which direction must Steve head to reach the finish line?

EXAMPLE 1 Solving a problem by using the sine and cosine laws

Determine the direction in which Steve should head to reach the finish line.

Liz's Solution



I drew a sketch and labelled the interior angle I needed to calculate as θ.

$$\angle J = 5° + 22°$$
$$= 27°$$

A bearing of 355° means that Steve's destination should have been 5° west of north. But he drifted 22° east of north. So I added 5° and 22° to get $\angle J$.

$$s = 40 \text{ km}$$
$$t = 32 \text{ km}$$
$$j^2 = s^2 + t^2 - 2st\cos J$$
$$j^2 = 40^2 + 32^2 - 2(40)(32)\cos 27°$$

I didn't have information to be able to use the sine law to determine angle θ. So I used the cosine law to first calculate side j.

$$j = \sqrt{40^2 + 32^2 - 2(40)(32)\cos 27°}$$

To solve for j, I took the square root of both sides of the equation.

$$j \doteq 18.521 \text{ km}$$

I used a calculator to evaluate.

$$\frac{\sin \theta}{40} = \frac{\sin 27°}{18.521}$$

I now had information to be able to use the sine law.

$$\cancel{40}^{1} \times \frac{\sin \theta}{\cancel{40}_{1}} = 40 \times \frac{\sin 27°}{18.521}$$

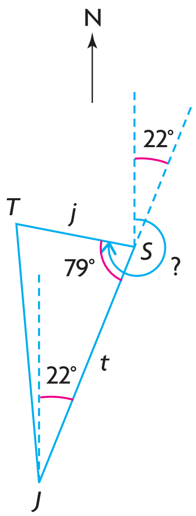
$$\sin \theta = 40 \times \frac{\sin 27°}{18.521}$$

To solve for angle θ, I multiplied both sides of the equation by 40.

$$\theta = \sin^{-1}\left(40 \times \frac{\sin 27°}{18.521}\right)$$

$$\theta \doteq 79°$$

I used the inverse sine function on a calculator to evaluate.



I now knew angle θ, but I needed to determine the bearing to T from S. I sketched the triangle with side t extended. I labelled the angles I knew and the angle I needed to determine.

$$\text{bearing} = 22° + 180° + 79°$$
$$= 281°$$

The bearing is measured from due north. So I added all the angles from due north to side j. I knew that the angle along the extended side t is 180°.

Steve must head at a bearing of about 281° to reach the finish line.

Reflecting

A. Why didn't Liz first use the sine law in her solution?

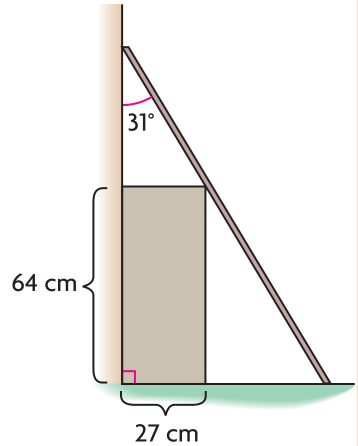
B. How is a diagram of the situation helpful? Explain.

C. Is it possible to solve this problem *without* using the sine law or the cosine law? Justify your answer.

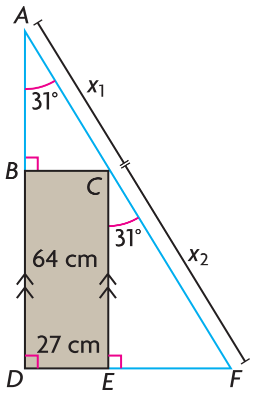
APPLY the Math

EXAMPLE 2 Solving a problem by using primary trigonometric ratios

A ladder leaning against a wall makes an angle of 31° with the wall. The ladder just touches a box that is flush against the wall and the ground. The box has a height of 64 cm and a width of 27 cm. How long, to the nearest centimetre, is the ladder?



Denis's Solution



The box and ladder form two right triangles. I labelled the two parts of the ladder as x_1 and x_2. The total length of the ladder is the sum of x_1 and x_2.

Since CE is parallel to AD, $\angle ECF$ is equal to $\angle BAC$, which is 31°.

In $\triangle ABC$:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 31° = \frac{BC}{x_1}$$

$$\sin 31° = \frac{27}{x_1}$$

$$x_1 \times \sin 31° = \overset{1}{\cancel{x_1}} \times \frac{27}{\underset{1}{\cancel{x_1}}}$$

$$x_1 \times \sin 31° = 27$$

In $\triangle CEF$:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 31° = \frac{CE}{x_2}$$

$$\cos 31° = \frac{64}{x_2}$$

$$x_2 \times \cos 31° = \overset{1}{\cancel{x_2}} \times \frac{64}{\underset{1}{\cancel{x_2}}}$$

$$x_2 \times \cos 31° = 64$$

In $\triangle ABC$, $BC = 27$ cm; in $\triangle CEF$, $CE = 64$ cm.

In each triangle, I knew a side and an angle and I needed to calculate the hypotenuse. So I used primary trigonometric ratios:

- sine, since BC is opposite the 31° angle
- cosine, since CE is adjacent to the 31° angle

To solve for x_1, I multiplied both sides of the equation by x_1 and divided both sides by sin 31°.

To solve for x_2, I multiplied both sides of the equation by x_2 and divided both sides by cos 31°.

$$\frac{x_1 \times \cancel{\sin 31°}^{1}}{\cancel{\sin 31°}_{1}} = \frac{27}{\sin 31°} \qquad \frac{x_2 \times \cancel{\cos 31°}^{1}}{\cancel{\cos 31°}_{1}} = \frac{64}{\cos 31°}$$

$$x_1 = \frac{27}{\sin 31°} \qquad x_2 = \frac{64}{\cos 31°}$$

$$x_1 \doteq 52.423\,308\,71 \qquad x_2 \doteq 74.664\,537\,42$$

I used a calculator to evaluate.

$$\begin{aligned} \text{length} &= x_1 + x_2 \\ &= 52.423\,308\,71 + 74.664\,537\,42 \\ &\doteq 127 \text{ cm} \end{aligned}$$

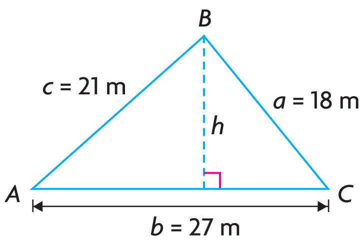
I added x_1 and x_2 to determine the length of the ladder.

The ladder is about 127 cm long.

EXAMPLE 3 Selecting a strategy to calculate the area of a triangle

Jim has a triangular backyard with side lengths of 27 m, 21 m, and 18 m. His bag of fertilizer covers 400 m^2. Does he have enough fertilizer?

Barbara's Solution



I drew a sketch with the longest side at the bottom. I labelled all the given information.

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$18^2 = 27^2 + 21^2 - 2(27)(21)\cos A$$
$$18^2 - 27^2 - 21^2 = -2(27)(21)\cos A$$

To determine the area, I needed the height, h, of the triangle. To determine h, I had to calculate an angle first, so I chose $\angle A$. I used the cosine law to determine $\angle A$.

$$\frac{18^2 - 27^2 - 21^2}{-2(27)(21)} = \left(\frac{\cancel{-2(27)(21)}^{1}}{\cancel{-2(27)(21)}_{1}}\right)\cos A$$

$$\cos A = \frac{18^2 - 27^2 - 21^2}{-2(27)(21)}$$

To solve for $\angle A$, I divided both sides of the equation by $-2(27)(21)$.

$$\angle A = \cos^{-1}\left(\frac{18^2 - 27^2 - 21^2}{-2(27)(21)}\right)$$

$$\angle A \doteq 41.75°$$

I used the inverse cosine function on a calculator to evaluate.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$

Since I knew $\angle A$, I used a primary trigonometric ratio to calculate h.

$$\sin A = \frac{h}{21}$$

$$\sin 41.75° = \frac{h}{21}$$

$$21 \times \sin 41.75° = \cancel{21}^{1} \times \frac{h}{\cancel{21}_{1}}$$

To solve for h, I multiplied both sides of the equation by 21.

$$21 \times \sin 41.75° = h$$

$$13.98 \text{ m} \doteq h$$

I used a calculator to evaluate.

$$A = \frac{1}{2} b \times h$$

$$= \left(\frac{1}{2}\right) 27 \times 13.98$$

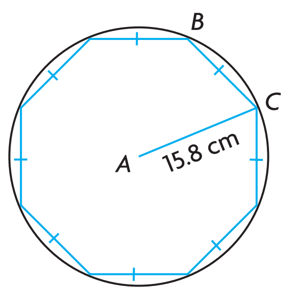
$$\doteq 189 \text{ m}^2$$

I substituted the values for b and h into the formula for the area of a triangle.

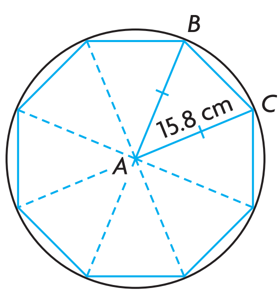
Since 189 is less than half of 400, Jim has enough fertilizer to cover his lawn twice and still have some fertilizer left over.

EXAMPLE 4 Solving a problem to determine a perimeter

A regular octagon is inscribed in a circle of radius 15.8 cm. What is the perimeter, to the nearest tenth of a centimetre, of the octagon?



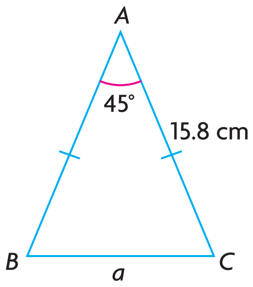
Shelley's Solution



An octagon is made up of eight identical triangles, each of which is isosceles because two sides are the same length (radii of the circle).

$$\angle A = 360° \div 8$$
$$= 45°$$

A circle measures 360°. $\angle A$ is $\frac{1}{8}$ of 360°. So I divided 360° by 8 to get $\angle A$.



I labelled the base of $\triangle BAC$ as a.

$$\angle A + \angle B + \angle C = 180°$$
$$45° + 2\angle B = 180°$$
$$2\angle B = 180° - 45°$$
$$2\angle B = 135°$$
$$\angle B = 67.5°$$

Since $\triangle BAC$ is isosceles, $\angle B = \angle C$. The sum of the interior angles is 180°, so I calculated $\angle B$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 45°} = \frac{15.8}{\sin 67.5°}$$

I knew an angle and the side opposite that angle. So I used the sine law to calculate a.

$$\overset{1}{\cancel{\sin 45°}} \times \frac{a}{\underset{1}{\cancel{\sin 45°}}} = \sin 45° \times \frac{15.8}{\sin 67.5°}$$

To solve for a, I multiplied both sides of the equation by sin 45°.

$$a = \sin 45° \times \frac{15.8}{\sin 67.5°}$$
$$a \doteq 12.09 \text{ cm}$$

I used a calculator to evaluate.

$$\text{perimeter} = 12.09 \times 8$$
$$\doteq 96.7 \text{ cm}$$

The octagon is made up of eight sides of length equal to a. So I multiplied a by 8 to get the perimeter.

The perimeter of the octagon is about 96.7 cm.

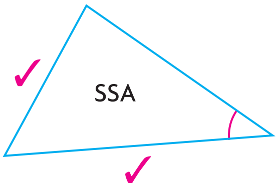
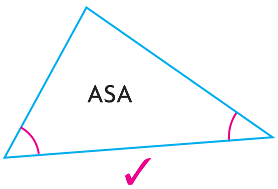
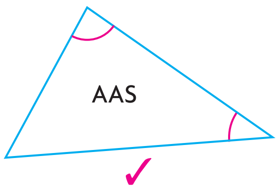
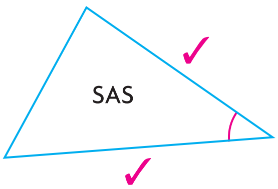
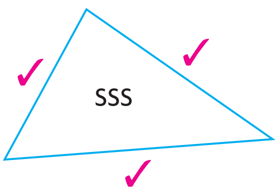
In Summary

Key Idea

- The primary trigonometric ratios, the sine law, and the cosine law can be used to solve problems involving triangles. The method you use depends on the information you know about the triangle and what you want to determine.

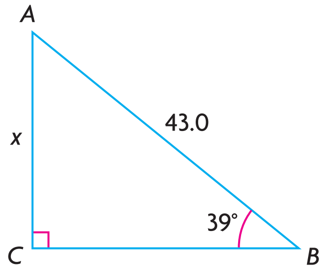
Need to Know

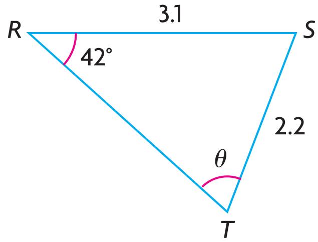
- If the triangle in the problem is a right triangle, use the primary trigonometric ratios.
- If the triangle is oblique, use the sine law and/or the cosine law.

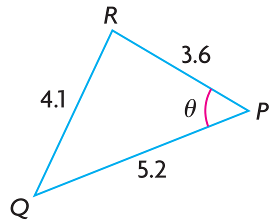
Given	Required to Find	Use
	angle	sine law
 or 	side	sine law
	side	cosine law
	angle	cosine law

CHECK Your Understanding

1. For each triangle, describe how you would solve for the unknown side or angle.

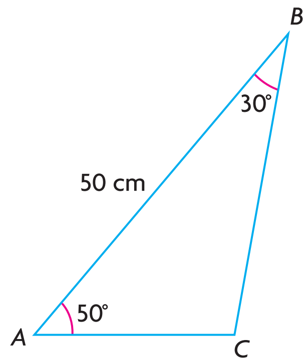
a) 

b) 

c) 

2. Complete a solution for each part of question 1. Round x to the nearest tenth of a unit and θ to the nearest degree.

PRACTISING



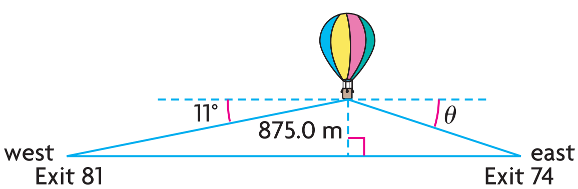
3. Determine the area of $\triangle ABC$, shown at the right, to the nearest square centimetre.
K

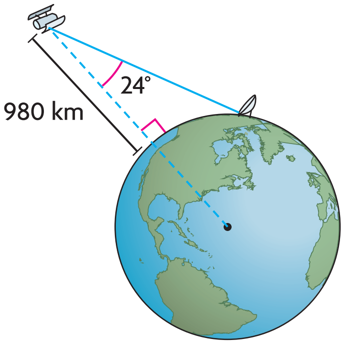
4. The legs of a collapsible stepladder are each 2.0 m long. What is the maximum distance between the front and rear legs if the maximum angle at the top is 40°? Round your answer to the nearest tenth of a metre.

5. To get around an obstacle, a local electrical utility must lay two sections of underground cable that are 371.0 m and 440.0 m long. The two sections meet at an angle of 85°. How much extra cable is necessary due to the obstacle? Round your answer to the nearest tenth of a metre.
A

6. A surveyor is surveying three locations (M, N, and P) for new rides in an amusement park around an artificial lake. $\angle MNP$ is measured as 57°. MN is 728.0 m and MP is 638.0 m. What is the angle at M to the nearest degree?

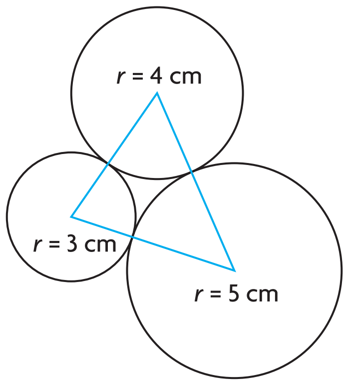
7. Mike's hot-air balloon is 875.0 m directly above a highway. When he is looking west, the angle of depression to Exit 81 is 11°. The exit numbers on this highway represent the number of kilometres left before the highway ends. What is the angle of depression, to the nearest degree, to Exit 74 in the east?

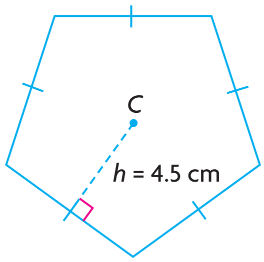




8. A satellite is orbiting Earth 980 km above Earth's surface. A receiving dish is located on Earth such that the line from the satellite to the dish and the line from the satellite to Earth's centre form an angle of 24° as shown at the left. If a signal from the satellite travels at 3×10^8 m/s, how long does it take to reach the dish? Round your answer to the nearest thousandth of a second.

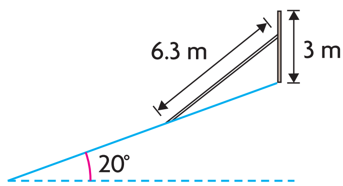
9. Three circles with radii of 3 cm, 4 cm, and 5 cm are touching each other as shown. A triangle is drawn connecting the three centres. Calculate all the interior angles of the triangle. Round your answers to the nearest degree.

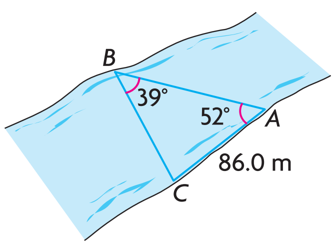




10. T Given the regular pentagon shown at the left, determine its perimeter to the nearest tenth of a centimetre and its area to the nearest tenth of a square centimetre.

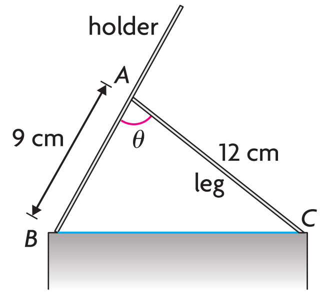
11. A 3 m high fence is on the side of a hill and tends to lean over. The hill is inclined at an angle of 20° to the horizontal. A 6.3 m brace is to be installed to prop up the fence. It will be attached to the fence at a height of 2.5 m and will be staked downhill from the base of the fence. What angle, to the nearest degree, does the brace make with the hill?





12. A surveyor wants to calculate the distance *BC* across a river. He selects a position, *A*, so that *CA* is 86.0 m. He measures ∠*ABC* and ∠*BAC* as 39° and 52°, respectively, as shown at the left. Calculate the distance *BC* to the nearest tenth of a metre.

13. For best viewing, a document holder for people who work at computers should be inclined between $61°$ and $65°$ $(\angle ABC)$. A 12 cm support leg is attached to the holder 9 cm from the bottom. Calculate the minimum and maximum angle θ that the leg must make with the holder. Round your answers to the nearest degree.

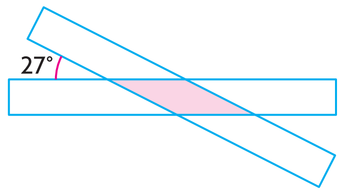


14. **C** Match each method with a problem that can be solved by that method. Describe how each method could be used to complete a solution.

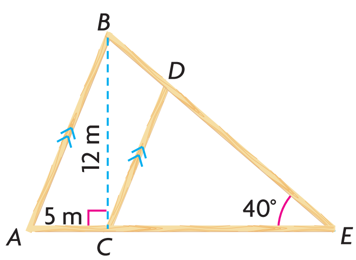
Method	Problems
Cosine law	Chris lives in a U-shaped building. From his window, he sights Bethany's window at a bearing of 328° and Josef's window at a bearing of 19°. Josef's window is 54 m from Bethany's and both windows are directly opposite each other. How far is each window from Chris's window?
Sine law	When the Sun is at an angle of elevation of 41°, Martina's treehouse casts a shadow that is 11.4 m long. Assuming that the ground is level, how tall is Martina's treehouse?
Primary trigonometric ratios	Ken walks 3.8 km west and then turns clockwise 65° before walking another 1.7 km. How far does Ken have to walk to get back to where he started?

Extending

15. Two paper strips, each 2.5 cm wide, are laying across each other at an angle of $27°$, as shown at the right. What is the area of the overlapping paper? Round your answer to the nearest tenth of a square centimetre.



16. The diagram shows a roofing truss with AB parallel to CD. Calculate the total length of wood needed to construct the truss. Round your answer to the nearest metre.



17. Lucas takes a 10.0 m rope and creates a triangle with interior angles of $30°$, $70°$, and $80°$. How long, to the nearest tenth of a metre, is each side?