5.4 Investigating and Applying the Cosine Law in Acute Triangles

YOU WILL NEED

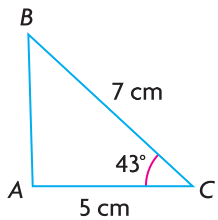
- dynamic geometry software

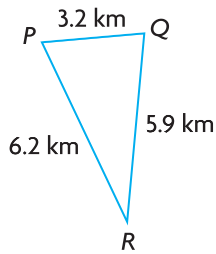
GOAL

Verify the cosine law and use it to solve real-life problems.

INVESTIGATE the Math

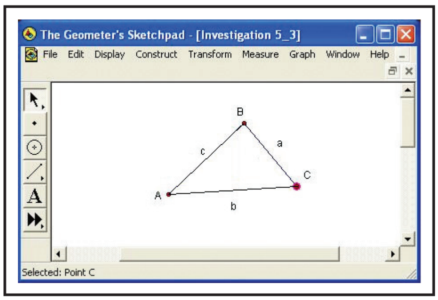
The sine law is useful only when you know two sides and an angle opposite one of those sides or when you know two angles and a side. But in the triangles shown, you don't know this information, so the sine law cannot be applied.





? How can the Pythagorean theorem be modified to relate the sides and angles in these types of triangles?

A. Use dynamic geometry software to construct any acute triangle.



B. Label the vertices A, B, and C. Then name the sides a, b, and c as shown.

C. Measure all three interior angles and all three sides.

D. Drag vertex C until $\angle C = 90°$.

E. What is the Pythagorean relationship for this triangle? Using the measures of a and b, choose **Calculate ...** from the **Measure** menu to determine $a^2 + b^2$. Use the measure of c and repeat to determine c^2.

Tech Support

For help using dynamic geometry software, see Technical Appendix, B-21 and B-22.

F. Does $a^2 + b^2 = c^2$? If not, are they close? Why might they be off by a little bit?

G. Move vertex C farther away from AB to create an acute triangle. How does the value of $a^2 + b^2$ compare with that of c^2?

H. Using the measures of $a^2 + b^2$ and c^2, choose **Calculate ...** from the **Measure** menu to determine $a^2 + b^2 - c^2$. How far off is this value from the Pythagorean theorem? Copy the table shown and record your observations.

Triangle	a	b	c	$\angle C$	$a^2 + b^2$	c^2	$a^2 + b^2 - c^2$
1							
2							
3							
4							
5							

I. Move vertex C to four other locations and record your observations. Make sure that one of the triangles has $\angle C = 90°$, while the rest are acute.

J. Use a calculator to determine $2ab \cos C$ for each of your triangles. Add this column to your table and record these values. How do they compare with the values of $a^2 + b^2 - c^2$?

K. Based on your observations, modify the Pythagorean theorem to relate c^2 to $a^2 + b^2$.

Reflecting

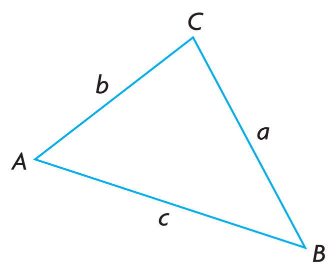
L. How is the Pythagorean theorem a special case of the relationship you found? Explain.

M. Based on your observations, how did the value of $\angle C$ affect the value of $a^2 + b^2 - c^2$?

N. Explain how you would use the cosine law to relate each pair of values in any acute triangle.
- a^2 to $b^2 + c^2$
- b^2 to $a^2 + c^2$

cosine law

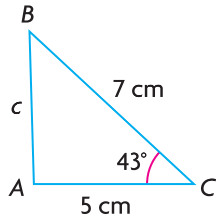
in any acute $\triangle ABC$,
$c^2 = a^2 + b^2 - 2ab \cos C$



APPLY the Math

EXAMPLE 1 Selecting the cosine law as a strategy to calculate an unknown length

Determine the length of c to the nearest centimetre.



Anita's Solution

$a = 7$ cm

$b = 5$ cm

$\angle C = 43°$

I can't be certain that the triangle has a 90° angle, and I can't use the sine law with the given information.

$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 7^2 + 5^2 - 2(7)(5)\cos 43°$

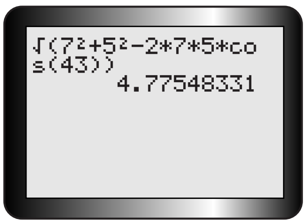
I chose the cosine law since I knew two sides and the **contained angle**, $\angle C$.

contained angle

the acute angle between two known sides

$c = \sqrt{7^2 + 5^2 - 2(7)(5)\cos 43°}$

To solve for c, I took the square root of both sides of the equation.



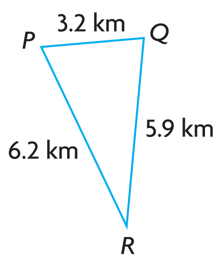
I used a calculator to evaluate.

$c \doteq 5$ cm

Side c is about 5 cm long.

EXAMPLE 2 Selecting the cosine law as a strategy to calculate an unknown angle

Determine the measure of $\angle R$ to the nearest degree.



Kew's Solution

$p = 5.9$ km

$q = 6.2$ km

$r = 3.2$ km

I knew three sides and no angles.

$$r^2 = p^2 + q^2 - 2pq\cos R$$

$$3.2^2 = 5.9^2 + 6.2^2 - 2(5.9)(6.2)\cos R$$

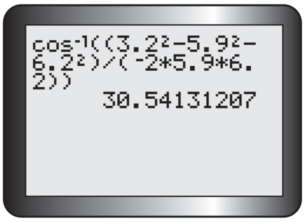
I chose the cosine law. I wrote the formula in terms of $\angle R$.

$$3.2^2 - 5.9^2 - 6.2^2 = -2(5.9)(6.2)\cos R$$

$$\frac{3.2^2 - 5.9^2 - 6.2^2}{-2(5.9)(6.2)} = \left(\frac{\overset{1}{\cancel{-2(5.9)(6.2)}}}{\underset{1}{\cancel{-2(5.9)(6.2)}}}\right)\cos R$$

To solve for $\angle R$, I divided both sides of the equation by $-2(5.9)(6.2)$.

$$\frac{3.2^2 - 5.9^2 - 6.2^2}{-2(5.9)(6.2)} = \cos R$$



I used the inverse cosine function on a calculator to evaluate.

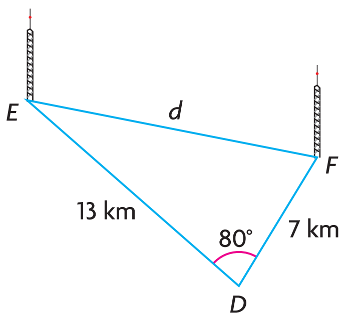
$$\angle R \doteq 31°$$

The measure of $\angle R$ is about 31°.

EXAMPLE 3 Solving a problem by using the cosine law

Ken's cell phone detects two transmission antennas, one 7 km away and the other 13 km away. From his position, the two antennas appear to be separated by an angle of 80°. How far apart, to the nearest kilometre, are the two antennas?

Chantal's Solution



I drew a sketch and labelled the distance between the two antennas as *d*.

$e = 7$ km

$f = 13$ km

$\angle D = 80°$

The triangle doesn't have a 90° angle. I knew two sides and the contained angle, so I chose the cosine law.

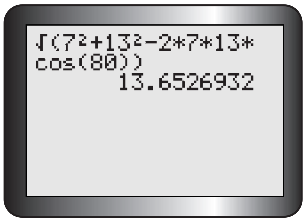
$d^2 = e^2 + f^2 - 2ef \cos D$

I wrote the formula in terms of $\angle D$.

$d^2 = 7^2 + 13^2 - 2(7)(13)\cos 80°$

$d = \sqrt{7^2 + 13^2 - 2(7)(13)\cos 80°}$

To solve for d, I took the square root of both sides of the equation.



I used a calculator to evaluate.

$d \doteq 14$ km

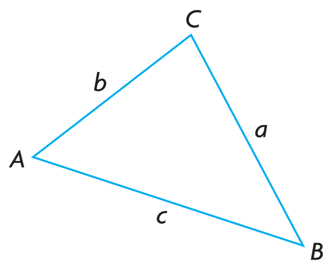
The two antennas are about 14 km apart.

In Summary

Key Idea

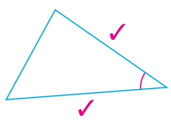
- The cosine law states that in any $\triangle ABC$,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

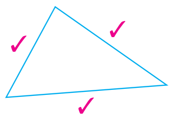


Need to Know

- The cosine law can be used only when you know
  - two sides and the contained angle or



  - all three sides



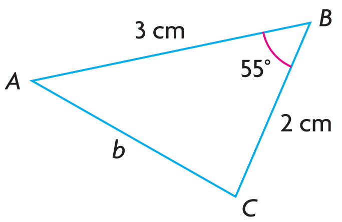
- The contained angle in a triangle is the angle between two known sides.

CHECK Your Understanding

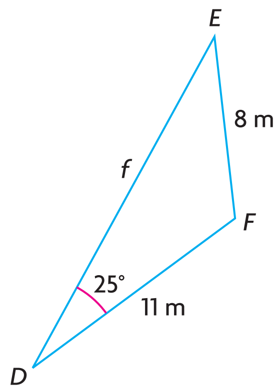
1. **i)** In which triangle is it necessary to use the cosine law to calculate the third side? Justify your answer.

ii) State the formula you would use to determine the length of side b in $\triangle ABC$.

a)

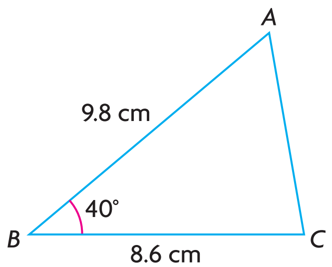


b)



2. Determine the length of each unknown side to the nearest tenth of a centimetre.

a)



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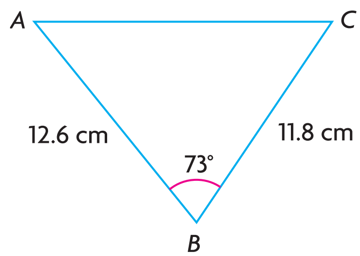
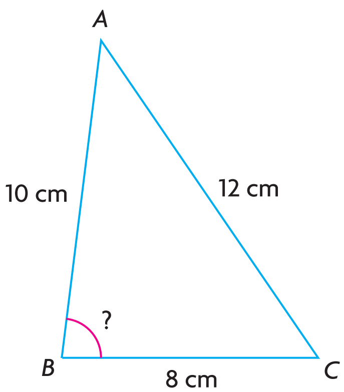


3. Determine each indicated angle to the nearest degree.

a)



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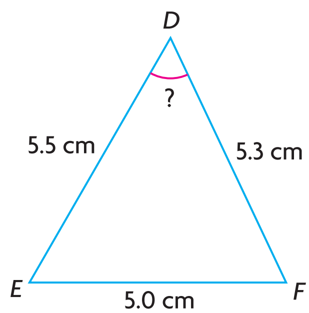
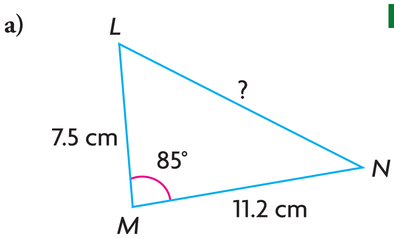


PRACTISING

4. Determine each indicated length to the nearest tenth of a centimetre.
K

a)



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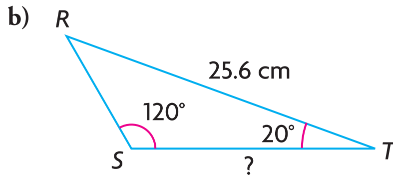
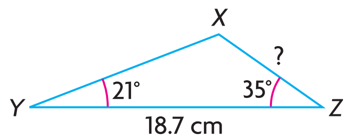


c)



5. Solve each triangle. Round each length to the nearest tenth of a centimetre and each angle to the nearest degree.

a) $\triangle ABC$: $\angle A = 68°$, $b = 10.1$ cm, $c = 11.1$ cm

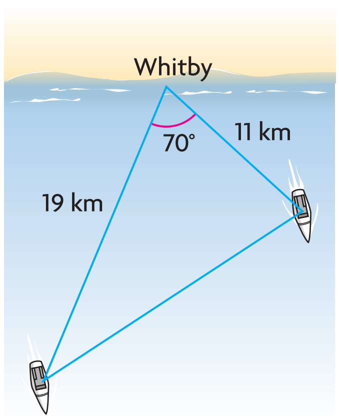
b) $\triangle DEF$: $\angle D = 52°$, $e = 7.2$ cm, $f = 9.6$ cm

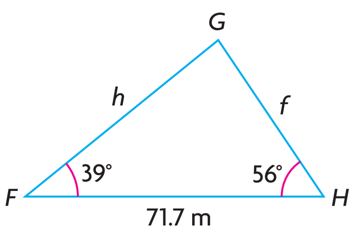
c) $\triangle HIF$: $\angle H = 35°$, $i = 9.3$ cm, $f = 12.5$ cm

d) $\triangle PQR$: $p = 7.5$ cm, $q = 8.1$ cm, $r = 12.2$ cm

6. A triangle has sides that measure 5 cm, 6 cm, and 10 cm. Do any of the angles in this triangle equal 30°? Explain.

7. Two boats leave Whitby harbour at the same time. One boat heads 19 km to its destination in Lake Ontario. The second boat heads on a course 70° from the first boat and travels 11 km to its destination. How far apart, to the nearest kilometre, are the boats when they reach their destinations?
A

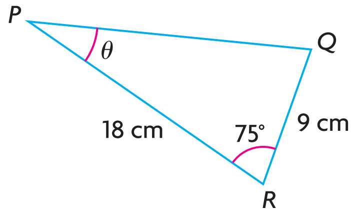




8. Louis says that he has no information to use the sine law to solve $\triangle FGH$ shown at the left and that he must use the cosine law instead. Is he correct? Describe how you would solve for each unknown side and angle.

9. Driving a snowmobile across a frozen lake, Sheldon starts from the most westerly point and travels 8.0 km before he turns right at an angle of 59° and travels 6.1 km, stopping at the most easterly point of the lake. How wide, to the nearest tenth of a kilometre, is the lake?

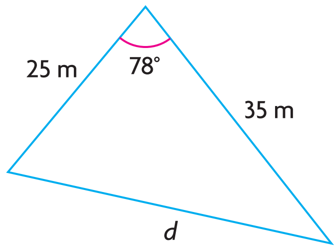
10. What strategy would you use to calculate angle θ in $\triangle PQR$ shown at the right? Justify your choice and then use your strategy to solve for θ to the nearest degree.



11. **T** A clock with a radius of 15 cm has an 11 cm minute hand and a 7 cm hour hand. How far apart, to the nearest centimetre, are the tips of the hands at each time?

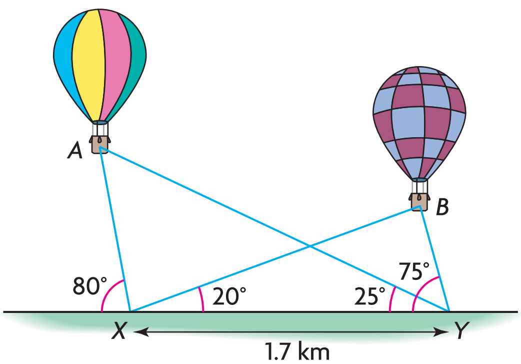
a) 3:30 p.m. **b)** 6:38 a.m.

12. **C** Use the triangle shown at the right to create a problem that involves its side lengths and interior angles. Then describe how to determine d.



Extending

13. Two observers standing at points X and Y are 1.7 km apart. Each person measures angles of elevation to two balloons, A and B, flying overhead as shown. For each question, round your answer to the nearest tenth of a kilometre.



a) How far is balloon A from point X? From point Y?
b) How far is balloon B from point X? From point Y?
c) How far apart are balloons A and B?