

5.3

Investigating and Applying the Sine Law in Acute Triangles

GOAL

Verify the sine law and use it to solve real-life problems.

INVESTIGATE the Math

The trigonometric ratios sine, cosine, and tangent are defined only for right triangles. In an **oblique triangle**, these ratios no longer apply.

? In an oblique triangle, what is the relationship between a side and the sine of the angle opposite that side?

- Use dynamic geometry software to construct any acute triangle.
- Label the vertices A , B , and C . Then name the sides a , b , and c as shown.
- Measure all three interior angles and all three sides.
- Using one side length and the angle opposite that side, choose **Calculate ...** from the **Measure** menu to evaluate the ratio $\frac{\text{side length}}{\sin(\text{opposite angle})}$.
- Repeat part D for the other sides and angles. What do you notice?
- Drag any vertex of your triangle. What happens to the sides, angles, and ratios? Now drag the other two vertices and explain.
- Express your findings
 - in words
 - with a mathematical relationship

Reflecting

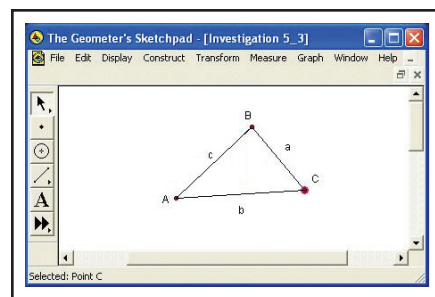
- Are the reciprocals of the ratios you found equal? Explain how you know.
- The relationship that you verified is the **sine law**. Why is this name appropriate?
- Does the sine law apply to all types of triangles (obtuse, acute, and right)? Explain how you know.

YOU WILL NEED

- dynamic geometry software

oblique triangle

a triangle (acute or obtuse) that does not contain a right angle

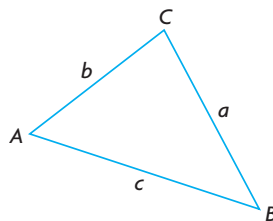


Tech Support

For help using dynamic geometry software, see Technical Appendix, B-21 and B-22.

sine law

in any acute triangle, the ratios of each side to the sine of its opposite angle are equal



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ OR}$$

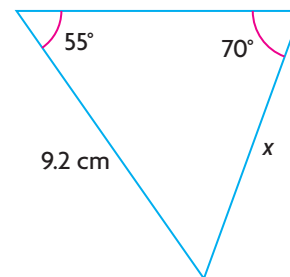
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

APPLY the Math

EXAMPLE 1

Using the sine law to calculate an unknown length

Determine x to the nearest tenth of a centimetre.



Barbara's Solution

$$\frac{x}{\sin 55^\circ} = \frac{9.2}{\sin 70^\circ}$$

The triangle doesn't have a 90° angle. So it isn't convenient to use the primary trigonometric ratios.

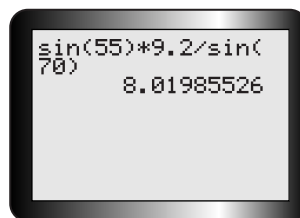
I chose the sine law because I knew two angles and the side opposite one of those angles.

I wrote the ratios with the sides in the numerators to make the calculations easier.

$$\sin 55^\circ \times \frac{x}{\sin 55^\circ} = \sin 55^\circ \times \frac{9.2}{\sin 70^\circ}$$

To solve for x , I multiplied both sides of the equation by $\sin 55^\circ$.

$$x = \sin 55^\circ \times \frac{9.2}{\sin 70^\circ}$$



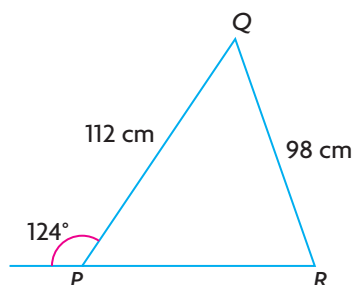
I used a calculator to evaluate.

$$x \doteq 8.0 \text{ cm}$$

The unknown side of the triangle is about 8.0 cm long.

EXAMPLE 2**Selecting the sine law as a strategy to calculate unknown angles**

Determine all the interior angles in $\triangle PQR$.
Round your answers to the nearest degree.

**Tom's Solution**

$$\begin{aligned}\angle QPR &= 180^\circ - 124^\circ \\ &= 56^\circ\end{aligned}$$

$\angle QPR$ and 124° add up to 180° . So, to determine $\angle QPR$, I subtracted 124° from 180° .

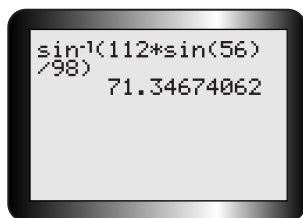
$$\frac{\sin R}{112} = \frac{\sin 56^\circ}{98}$$

$\triangle PQR$ doesn't have a 90° angle. So I chose the sine law, since I knew two sides and an angle opposite one of those sides. I wrote the ratios with the angles in the numerators to make it easier to solve for $\angle R$.

$$112 \times \frac{\sin R}{112} = 112 \times \frac{\sin 56^\circ}{98}$$

To solve for $\angle R$, I multiplied both sides of the equation by 112.

$$\sin R = 112 \times \frac{\sin 56^\circ}{98}$$



I used the inverse sine function on a calculator to evaluate.

$$\angle R \doteq 71^\circ$$

$$\begin{aligned}\angle Q &= 180^\circ - \angle QPR - \angle R \\ &= 180^\circ - 56^\circ - 71^\circ \\ &= 53^\circ\end{aligned}$$

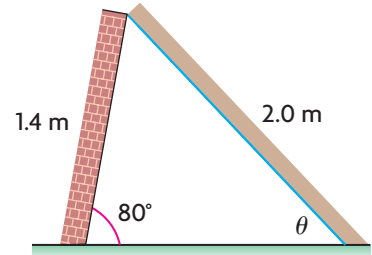
All three interior angles add up to 180° . So I subtracted $\angle R$ and $\angle QPR$ from 180° to determine $\angle Q$.

The interior angles in $\triangle PQR$ are 56° , 53° , and 71° .

EXAMPLE 3**Solving a problem by using the sine law**

A wall that is 1.4 m long has started to lean and now makes an angle of 80° with the ground. A 2.0 m board is jammed between the top of the wall and the ground to prop the wall up. Assume that the ground is level.

- What angle, to the nearest degree, does the board make with the ground?
- What angle, to the nearest degree, does the board make with the wall?
- How far, to the nearest tenth of a metre, is the board from the base of the wall?

**Isabelle's Solution**

$$\text{a) } \frac{\sin \theta}{1.4} = \frac{\sin 80^\circ}{2.0}$$

This triangle doesn't have a 90° angle. So I chose the sine law, since I knew two sides and an angle opposite one of those sides.

$$1.4 \times \frac{\sin \theta}{1.4} = 1.4 \times \frac{\sin 80^\circ}{2.0}$$

To solve for θ , I multiplied both sides of the equation by 1.4.

$$\sin \theta = 1.4 \times \frac{\sin 80^\circ}{2.0}$$

$$\theta = \sin^{-1} \left(1.4 \times \frac{\sin 80^\circ}{2.0} \right)$$

I used the inverse sine ratio on a calculator to determine the angle.

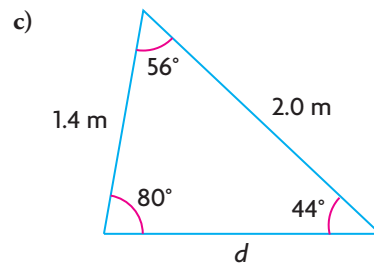
$$\theta \doteq 44^\circ$$

The board makes an angle of about 44° with the ground.

$$\text{b) } 180^\circ - 80^\circ - 44^\circ = 56^\circ$$

The interior angles add up to 180° . So I subtracted the two known angles from 180° .

The board makes an angle of about 56° with the wall.



I labelled the distance along the ground between the wall and the board as d .



$$\frac{d}{\sin 56^\circ} = \frac{2.0}{\sin 80^\circ}$$

Then I used the sine law.

$$\sin 56^\circ \times \frac{d}{\sin 56^\circ} = \sin 56^\circ \times \frac{2.0}{\sin 80^\circ}$$

To solve for d , I multiplied both sides of the equation by $\sin 56^\circ$ and used a calculator to evaluate.

$$d = \sin 56^\circ \times \frac{2.0}{\sin 80^\circ}$$

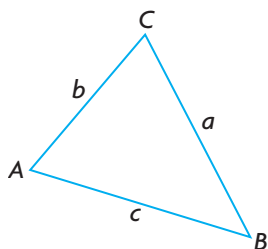
$$d \doteq 1.7 \text{ m}$$

The board is about 1.7 m from the base of the wall.

In Summary

Key Idea

- The sine law states that, in any acute $\triangle ABC$, the ratios of each side to the sine of its opposite angle are equal.



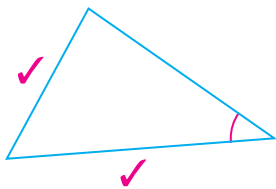
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

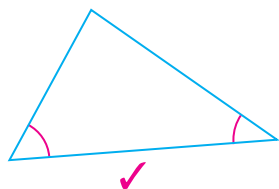
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Need to Know

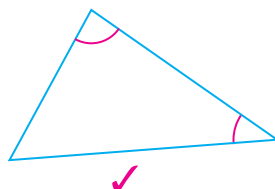
- The sine law can be used only when you know
 - two sides and the angle opposite a known side or



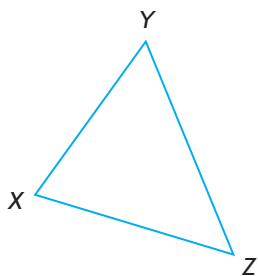
- two angles and any side



or



CHECK Your Understanding

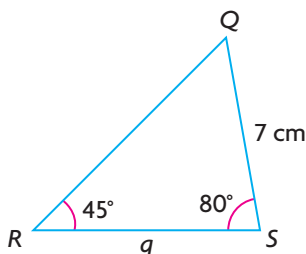
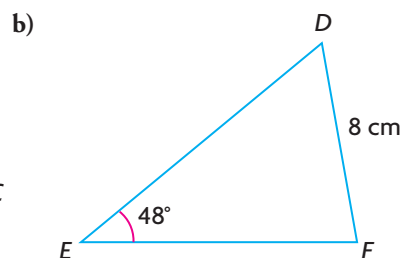
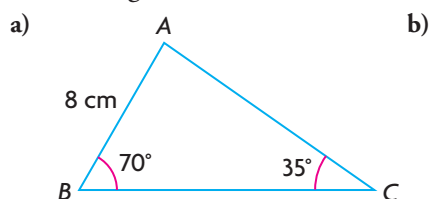


- Given $\triangle XYZ$ at the left, label the sides with lower case letters.
 - State the sine law for $\triangle XYZ$.
- Solve each equation. Round x to the nearest tenth of a unit and θ to the nearest degree.

a) $\frac{x}{\sin 22^\circ} = \frac{11.6}{\sin 71^\circ}$

b) $\frac{13.1}{\sin \theta} = \frac{29.2}{\sin 65^\circ}$

- Use the sine law to calculate b to the nearest centimetre and $\angle D$ to the nearest degree.

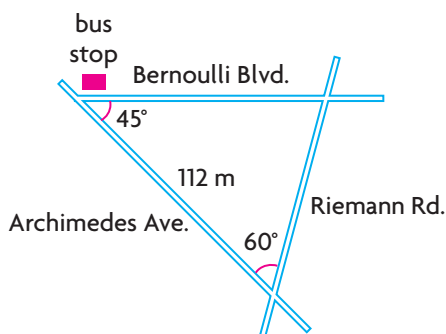


PRACTISING

- Given $\triangle RQS$ at the left, determine q to the nearest centimetre.

K

- Archimedes Avenue and Bernoulli Boulevard meet at an angle of 45° near a bus stop. Riemann Road intersects Archimedes Avenue at an angle of 60° . That intersection is 112 m from the bus stop.
 - At what angle do Riemann Road and Bernoulli Boulevard meet? Round your answer to the nearest degree.
 - How far, to the nearest metre, is the intersection of Riemann Road and Bernoulli Boulevard from the bus stop?



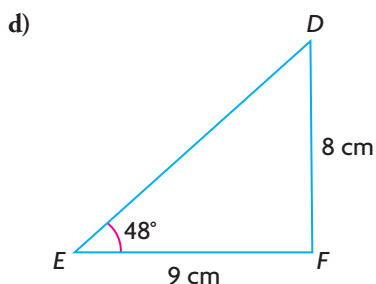
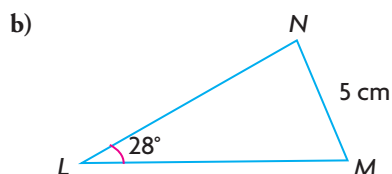
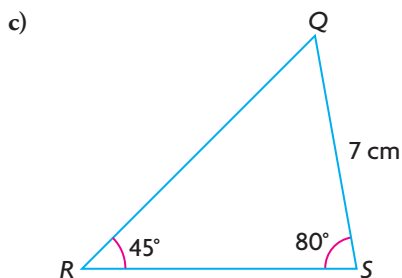
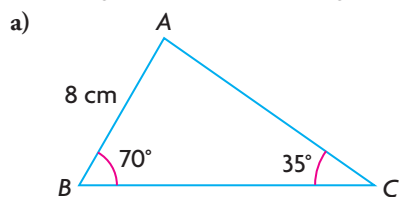
6. In $\triangle ABC$, two sides and an angle are given. Determine the value of $\angle C$ to the nearest degree and the length of b to the nearest tenth of a centimetre.

- $a = 2.4$ cm, $c = 3.2$ cm, $\angle A = 28^\circ$
- $a = 9.9$ cm, $c = 11.2$ cm, $\angle A = 58^\circ$
- $a = 8.6$ cm, $c = 9.4$ cm, $\angle A = 47^\circ$
- $a = 5.5$ cm, $c = 10.4$ cm, $\angle A = 30^\circ$

7. An isosceles triangle has two 5.5 cm sides and two 32° angles.

- Calculate the perimeter of the triangle to the nearest tenth of a centimetre.
- Calculate the area of the triangle to the nearest tenth of a square centimetre.

8. Solve each triangle. Round each length to the nearest centimetre and each angle to the nearest degree.

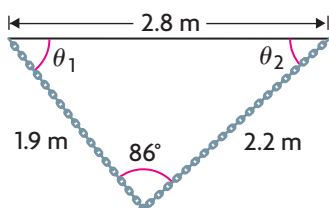


9. Solve each triangle. Round each length to the nearest tenth of a unit and each angle to the nearest degree.

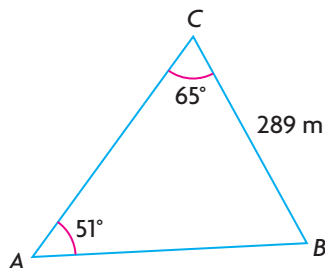
- $\triangle ABC$: $a = 10.3$, $c = 14.4$, $\angle C = 68^\circ$
- $\triangle DEF$: $\angle E = 38^\circ$, $\angle F = 48^\circ$, $f = 15.8$
- $\triangle GHJ$: $\angle G = 61^\circ$, $g = 5.3$, $j = 3.1$
- $\triangle KMN$: $k = 12.5$, $n = 9.6$, $\angle N = 42^\circ$
- $\triangle PQR$: $p = 1.2$, $r = 1.6$, $\angle R = 52^\circ$
- $\triangle XYZ$: $z = 6.8$, $\angle X = 42^\circ$, $\angle Y = 77^\circ$

Communication **Tip**

To solve a triangle, determine the measures of all unknown sides and angles.



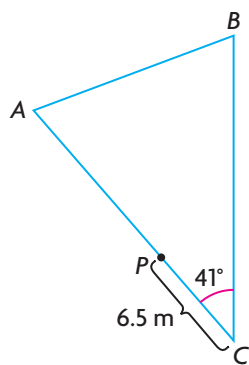
10. Toby uses chains and a winch to lift engines at his father's garage. Two hooks in the ceiling are 2.8 m apart. Each hook has a chain hanging from it. The chains are of length 1.9 m and 2.2 m. When the ends of the chains are attached, they form an angle of 86° . In this configuration, what acute angle, to the nearest degree, does each chain make with the ceiling?
11. Betsy installed cordless phones in the student centre at points A , B , and C as shown. Explain how you can use the given information to determine which two phones are farthest apart.



12. Tom says that he doesn't need to use the sine law because he can always determine a solution by using primary trigonometric ratios. Is Tom correct? What would you say to Tom to convince him to use the sine law to solve a problem?

Extending

13. Two angles in a triangle measure 54° and 38° . The longest side of the triangle is 24 cm longer than the shortest side. Calculate the length, to the nearest centimetre, of all three sides.



14. Use the sine law to show why the longest side of a triangle must be opposite the largest angle.
15. A triangular garden is enclosed by a fence. A dog is on a 5 m leash tethered to the fence at point P , 6.5 m from point C , as shown at the left. If $\angle ACB = 41^\circ$, calculate the total length, to the nearest tenth of a metre, of fence that the dog can reach.
16. Tara built a sculpture in the shape of a huge equilateral triangle of side length 4.0 m. Unfortunately, the ground underneath the sculpture was not stable, and one of the vertices of the triangle sank 55 cm into the ground. Assume that Tara's sculpture was built on level ground originally.
- What is the length, to the nearest tenth of a metre, of the two exposed sides of Tara's triangle now?
 - What percent of the triangle's surface area remains above ground? Round your answer to the nearest percent.