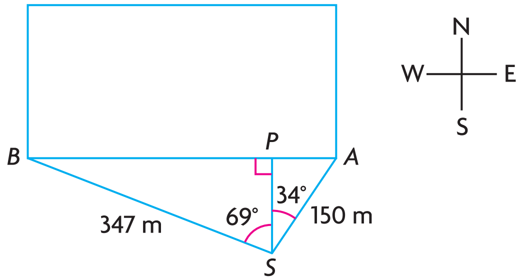
5.2 Solving Problems by Using Right-Triangle Models

GOAL

Solve real-life problems by using combinations of primary trigonometric ratios.

LEARN about the Math

A surveyor stands at point S and uses a laser transit and marking pole to sight two corners, A and B, on one side of a rectangular lot. The surveyor marks a reference point P on the line AB so that the line from P to S is perpendicular to AB. Corner A is 150 m away from S and 34° east of P. Corner B is 347 m away from S and 69° west of P.



Communication *Tip*

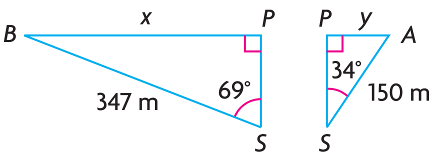
For the highest degree of accuracy, save intermediate answers by using the memory keys of your calculator. Round only after you do the very last calculation.

? How long, to the nearest metre, is the lot?

EXAMPLE 1 Selecting a strategy to determine a distance

Calculate the length of the building lot to the nearest metre.

Leila's Solution: Using the Sine Ratio



The length of the lot is $BP + PA$. I sketched the two right triangles that these sides formed. I labelled BP as x and PA as y.

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

In each triangle, I knew an angle and the hypotenuse. I needed to calculate the sides (x and y) opposite each given angle. So I used the sine ratio.

$$\sin 69° = \frac{x}{347} \qquad \sin 34° = \frac{y}{150}$$

$$347 \times \sin 69° = \overset{1}{\cancel{347}} \times \frac{x}{\underset{1}{\cancel{347}}} \qquad 150 \times \sin 34° = \overset{1}{\cancel{150}} \times \frac{y}{\underset{1}{\cancel{150}}}$$

To solve for x, I multiplied both sides of the first equation by 347.
To solve for y, I multiplied both sides of the second equation by 150.

$$x = 347 \times \sin 69° \qquad y = 150 \times \sin 34°$$

$$x \doteq 323.952\ 408 \qquad y \doteq 83.878\ 935\ 52$$

I used a calculator to evaluate.

$AB = BP + PA$

$= x + y$

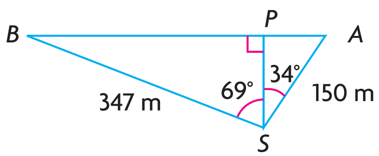
$= 323.952\,408 + 83.878\,935\,52$

$\doteq 408$ m

The lot is about 408 m long.

I added *x* and *y* to determine the length of the lot.

Tony's Solution: Using the Cosine Ratio and the Pythagorean Theorem



The length of the lot is *BP* + *PA*. *PS* is a shared side of △*BPS* and △*APS*. If I knew *PS*, then I could use the Pythagorean theorem to determine *BP* and *PA*.

In $\triangle APS$:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 34° = \frac{PS}{150}$$

AS is the hypotenuse and *PS* is adjacent to the 34° angle. So, to determine *PS*, I used the cosine ratio.

$$150 \times \cos 34° = \overset{1}{\cancel{150}} \times \frac{PS}{\underset{1}{\cancel{150}}}$$

$$PS = 150 \times \cos 34°$$

$$PS \doteq 124.356 \text{ m}$$

To solve for *PS*, I multiplied both sides of the equation by 150 and used a calculator to evaluate.

In $\triangle APS$:

$AS^2 = PS^2 + PA^2$

$150^2 = 124.356^2 + PA^2$

$PA^2 = 150^2 - 124.356^2$

$PA^2 = 7036$

$PA = \sqrt{7036}$

$PA \doteq 83.880\,867\,9$ m

In $\triangle BPS$:

$BS^2 = PS^2 + BP^2$

$347^2 = 124.356^2 + BP^2$

$BP^2 = 347^2 - 124.356^2$

$BP^2 = 104\,945$

$BP = \sqrt{104\,945}$

$BP \doteq 323.952\,157$ m

Next, I used the Pythagorean theorem to determine *PA* and *BP* .

$AB = BP + PA$

$= 83.880\,867\,9 + 323.952\,157$

$\doteq 408$ m

The lot is about 408 m long.

I added *BP* and *PA* to determine the length of the lot.

Reflecting

A. Explain why the length of the lot cannot be determined directly using either the Pythagorean theorem or the primary trigonometric ratios.

B. How are Leila's and Tony's solutions the same? How are they different?

C. Why did the surveyor choose point P so that PS would be perpendicular to AB?

APPLY the Math

EXAMPLE 2 Using trigonometric ratios to calculate angles

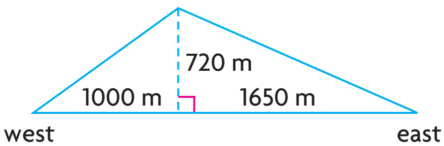
Karen is a photographer taking pictures of the Burlington Skyway bridge. She is in a helicopter hovering 720 m above the bridge, exactly 1 km horizontally from the west end of the bridge. The Skyway spans a distance of 2650 m from east to west.

a) From Karen's position, what are the angles of depression, to the nearest degree, of the east and west ends of the bridge?

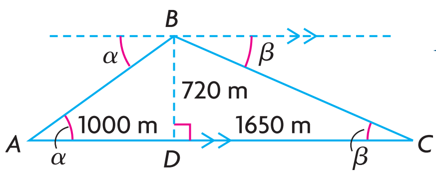
b) If Karen's camera has a wide-angle lens that can capture 150°, can she get the whole bridge in one shot?

Ali's Solution

a)



I sketched the position of the helicopter relative to the bridge. The horizontal distance from the helicopter to the west end of the bridge is 1000 m. So I subtracted 1000 from 2650 to determine the horizontal distance from the helicopter to the east end.



I drew a dashed line parallel to the base of the triangle and labelled the angles of depression as α and β. In $\triangle ABD$, I knew that $\angle BAD = \alpha$ because $\angle BAD$ and α are alternate angles between parallel lines. By the same reasoning, $\angle BCD = \beta$.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

I knew the opposite and adjacent sides in each right triangle. So, to determine α and β, I used the tangent ratio.

$$\tan\alpha = \frac{720}{1000} \qquad \tan\beta = \frac{720}{1650}$$

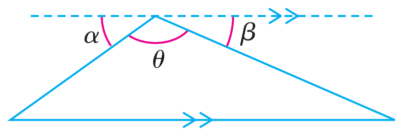
$$\alpha = \tan^{-1}\left(\frac{720}{1000}\right) \qquad \beta = \tan^{-1}\left(\frac{720}{1650}\right)$$

$$\alpha \doteq 35.8° \qquad \beta \doteq 23.6°$$

I used the inverse tangent function on my calculator to evaluate each angle.

The angles of depression are about 36° (west end) and about 24° (east end).

b)



To determine whether Karen can get the whole bridge in one shot, I needed to know θ, the angle between the east and west ends of the bridge from Karen's perspective.

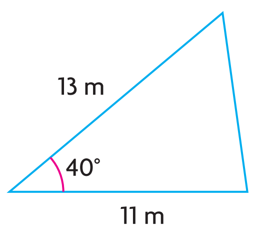
$$\begin{aligned}\theta &= 180° - \alpha - \beta\\ &= 180° - 35.8° - 23.6°\\ &= 120.6°\end{aligned}$$

I subtracted α and β from 180°.

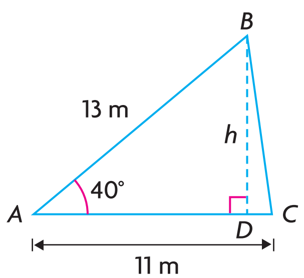
Since the angle from Karen's perspective is less than 150°, she will be able to get the whole bridge in one shot.

EXAMPLE 3 Using trigonometry to determine the area of a triangle

Elise's parents have a house with a triangular front lawn as shown. They want to cover the lawn with sod rather than plant grass seed. How much would it cost to put sod if it costs $13.75 per square metre?



Tina's Solution



I drew a dashed line for the height of the triangle and labelled it as h. The formula for the area of a triangle is $A = \frac{1}{2}b \times h$.

I knew the base, but I needed to calculate h before calculating the area.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

In right $\triangle ABD$, I knew the hypotenuse and the side adjacent to the 40° angle. The height is the side opposite the 40° angle, so I used the sine ratio.

$$\sin 40° = \frac{h}{13}$$

$$13 \times \sin 40° = \overset{1}{\cancel{13}} \times \frac{h}{\underset{1}{\cancel{13}}}$$

To solve for h, I multiplied both sides of the equation by 13 and used a calculator to evaluate.

$$h = 13 \times \sin 40°$$

$$h \doteq 8.356 \text{ m}$$

$$A = \frac{1}{2} b \times h$$

I substituted the base and height into the formula for the area of a triangle.

$$= \frac{1}{2}(11 \times 8.35)$$

$$\doteq 45.958 \text{ m}^2$$

$$\text{cost} = 45.958 \times 13.75$$

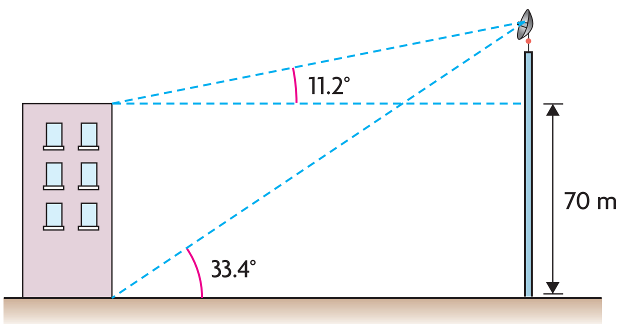
$$\doteq \$631.92$$

I multiplied the area by the cost per square metre to determine the cost to sod the lawn. I rounded to the nearest cent.

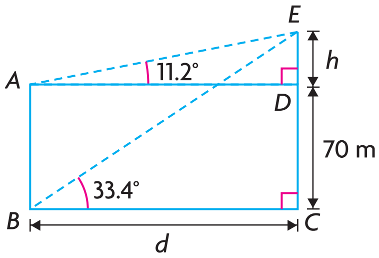
Elise's parents would have to spend \$631.92 to sod their lawn.

EXAMPLE 4 Solving a problem by using trigonometric ratios

A communications tower is some distance from the base of a 70 m high building. From the roof of the building, the angle of elevation to the top of the tower is 11.2°. From the base of the building, the angle of elevation to the top of the tower is 33.4°. Determine the height of the tower and how far it is from the base of the building. Round your answers to the nearest metre.



Pedro's Solution



I drew a sketch and labelled the distance between the tower and the building as *d*. The tower is more than 70 m tall, so I labelled the extra height above 70 m as *h*.

In $\triangle BEC$: In $\triangle AED$:

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 33.4° = \frac{h + 70}{d} \qquad \tan 11.2° = \frac{h}{d}$$

In $\triangle BEC$, *d* is adjacent to the 33.4° angle and the opposite side is 70 + *h*. In $\triangle AED$, *d* is adjacent to the 11.2° angle and the opposite side is *h*. Since I knew the adjacent and opposite sides in each right triangle, I used the tangent ratio.

$$d \times \tan 33.4° = \cancel{d} \times \frac{h + 70}{\cancel{d}} \qquad d \times \tan 11.2° = \cancel{d} \times \frac{h}{\cancel{d}}$$

$$d \times \tan 33.4° = h + 70 \qquad d \times \tan 11.2° = h$$

$$d = \frac{h + 70}{\tan 33.4°}$$

To solve for *d* in the first equation, I multiplied both sides by *d* and then divided both sides by tan 33.4°. To solve for *h* in the second equation, I multiplied both sides by *d*.

$$d = \frac{d \times \tan 11.2° + 70}{\tan 33.4°}$$

$$\tan 33.4° \times d = d \times \tan 11.2° + 70$$

$$d(\tan 33.4° - \tan 11.2°) = 70$$

$$0.461\,373\,177\,9d \doteq 70$$

$$d = \frac{70}{0.461\,373\,177\,9}$$

$$\doteq 151.721\,000\,2 \text{ m}$$

I substituted the second equation into the first and solved for *d*. I multiplied both sides of the equation by tan 33.4°. Then I simplified.

$$h = 151.721\,000\,2 \times \tan 11.2°$$

$$\doteq 30 \text{ m}$$

To determine *h*, I substituted the value of *d* into the second equation.

$$70 + 30 = 100 \text{ m}$$

To determine the height of the tower, I added 70 to *h*.

The tower is about 100 m tall and about 152 m from the base of the building.

In Summary

Key Idea

- If a situation involves calculating a length or an angle, try to represent the problem with a right-triangle model. If you can, solve the problem by using the primary trigonometric ratios.

Need to Know

- To calculate the area of a triangle, you can use the sine ratio to determine the height. For example, if you know a, b, and $\angle C$, then

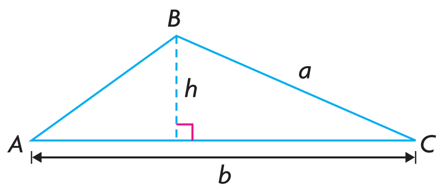
$$\sin C = \frac{h}{a}$$

$$h = a \sin C$$

The area of a triangle is

$$A = \frac{1}{2} b \times h$$

$$A = \frac{1}{2} b(a \sin C)$$



CHECK Your Understanding

1. Calculate the area of each triangle to the nearest tenth of a square centimetre.

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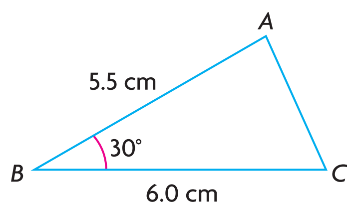
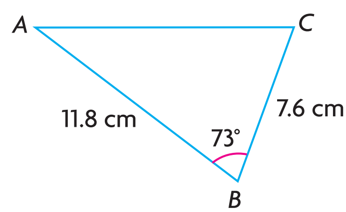
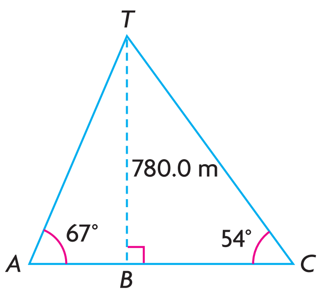


b)





2. A mountain is 780.0 m high. From points A and C, the angles of elevation to the top of the mountain are 67° and 54° as shown at the left. Explain how to calculate the length of a tunnel from A to C.

3. Karen and Anna are standing 23 m away from the base of a 23 m high house. Karen's eyes are 1.5 m above ground and Anna's eyes are 1.8 m above ground. Both girls observe the top of the house and measure its angle of elevation. Which girl will measure the greater angle of elevation? Justify your answer.

PRACTISING

4. The angle of elevation from the roof of a 10 m high building to the top of another building is 41°. The two buildings are 18 m apart at the base.
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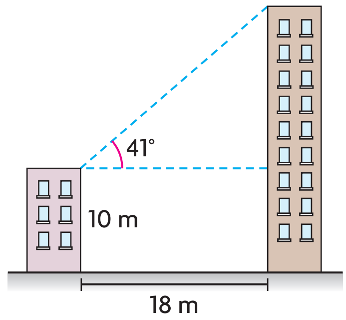
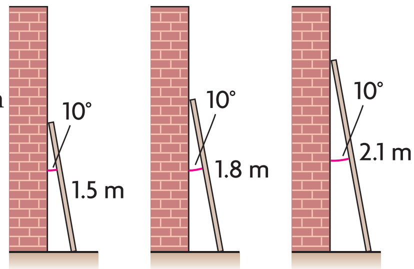
a) Which trigonometric ratio would you use to solve for the height of the taller building? Why?

b) How tall, to the nearest metre, is the taller building?



5. If the angle of elevation to the top of the pyramid of Cheops in Giza, Egypt, is 17.5°, measured 348 m from its base, can you calculate the height of the pyramid accurately? Explain your reasoning.



6. Darrin wants to lean planks of wood that are 1.5 m, 1.8 m, and 2.1 m long against the wall inside his garage. If the top of a plank forms an angle of less than 10° with the wall, the plank might fall over. If the bottom of a plank sticks out more than 30 cm from the wall, Darrin won't have room to park his car. Will Darrin be able to store all three planks in the garage? Justify your answer with calculations.

7. Jordan is standing on a bridge over the Welland Canal. His eyes are 5.1 m above the surface of the water. He sees a cargo ship heading straight toward him. From his position, the bow appears at an angle of depression of 5° and the stern appears at an angle of depression of 1°. For each question, round your answer to the nearest tenth of a metre.

a) What is the straight-line distance from the bow to Jordan?

b) What is the straight-line distance from the stern to Jordan?

c) What is the length of the ship from bow to stern?

8. A searchlight is mounted at the front of a helicopter flying 125 m above ground. The angle of depression of the light beam is 70°. An observer on the ground notices that the beam of light measures 5°. How wide, to the nearest metre, is *d*, the spot on the ground?
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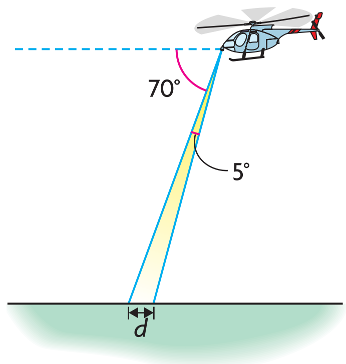
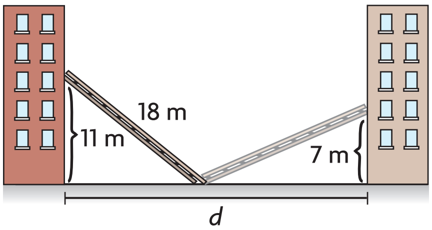
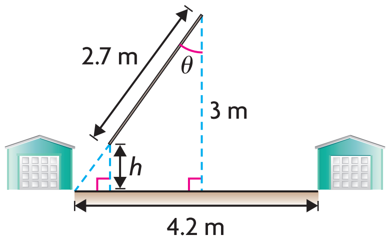




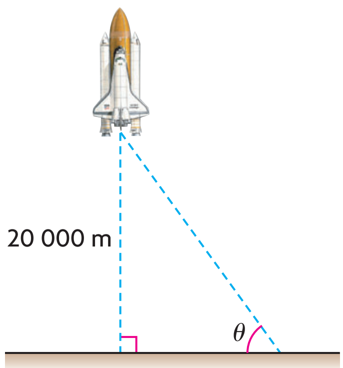
9. An 18 m long ladder is leaning against the wall of a building. The top of the ladder reaches a window 11 m above ground. If the ladder is tilted in the opposite direction, without moving its base, the top of the ladder can reach a window in another building that is 7 m above ground. How far apart, to the nearest metre, are the two buildings?

10. Kyle and Anand are standing on level ground on opposite sides of a tree. Kyle measures the angle of elevation to the treetop as 35°. Anand measures an angle of elevation of 30°. Kyle and Anand are 65 m apart. Kyle's eyes and Anand's eyes are 1.6 m above ground. How tall, to the nearest tenth of a metre, is the tree?

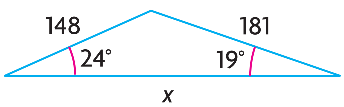


11. A tree branch 3 m above ground runs parallel to Noel's garage and his neighbour's. Noel wants to attach a rope swing on this branch. The garages are 4.2 m apart and the rope is 2.7 m long.
   a) What is the maximum angle, measured from the perpendicular, through which the rope can swing? Round your answer to the nearest degree.
   b) What is the maximum height above ground, to the nearest tenth of a metre, of the end of the rope?

12. A regular hexagon has a perimeter of 50 cm.
   a) Calculate the area of the hexagon to the nearest square centimetre.
   b) The hexagon is the base of a prism of height 100 cm. Calculate the volume, to the nearest cubic centimetre, and the surface area, to the nearest square centimetre, of the prism. Recall that volume = area of base × height. The surface area is the sum of the areas of all faces of the prism.



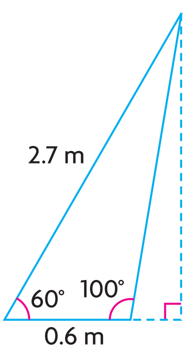
13. Lucien wants to photograph the separation of the solid rocket boosters from the space shuttle. He is standing 14 500 m from the launch pad, and the solid rocket boosters separate at 20 000 m above ground.
   a) Assume that the path of the shuttle launch is perfectly vertical up until the boosters separate. At what angle, to the nearest degree, should Lucien aim his camera?
   b) How far away is the space shuttle from Lucien at the moment the boosters separate? Round your answer to the nearest metre.



14. Sven wants to determine the unknown side of the triangle shown at the left. Even though it is not a right triangle, describe how Sven could use primary trigonometric ratios to determine *x*.

15. A pendulum that is 50 cm long is moved 40° from the vertical. What is the change in height from its initial position? Round your answer to the nearest centimetre.

Extending



16. Calculate the area of the triangle shown at the left to the nearest tenth of a square metre.